

Investigations of pre-breakdown phenomena: streamer discharges

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Physical principles of electrical breakdown are reviewed in this article. The phenomena of streamer formation and spontaneous branching are studied in terms of a fluid description based on kinetic theory. Particular attention is paid to a minimal model which is suitable for non-attaching gases. The evidence that anode directed fronts can branch due to a Laplacian instability is shown. Finally an electric shielding factor is introduced which allows us to extend previous results to curved geometries.

1. Introduction

Benjamin Franklin suspected that lightning was an electrical current in nature, and he wanted to see if he was right. One way to test his idea would be to see if the lightning would pass through metal. He decided to use a metal key and looked around for a way to get the key up near the lightning. He used a child's toy, a kite, to prove that lightning is really a stream of electrified air, known today as plasma. His famous stormy kite flight in June of 1752 led him to develop many of the terms that we still use today when we talk about electricity: battery, conductor, condenser, charge, discharge, uncharged, negative, minus, plus, electric shock and electrician.

A cubic centimetre of air contains roughly 2.7×10^{19} molecules of oxygen (O_2), nitrogen (N_2), water vapour (H_2O) and some other gases. The charged particles are bound by powerful electric forces to form electrically neutral atoms and molecules, and as a result the air is an excellent insulator. If we apply an electric field to a volume filled with neutral particles, electric current will not flow through the volume, because there are no charged particles present. However, if a strong electric field is applied to matter of low conductivity and some electrons or ions are created, then the few mobile charges can generate an avalanche of more charges by impact ionization. A low temperature plasma is created, resulting in an electric discharge. The change in the properties of a dielectric that causes it to become conductive is known as *electrical breakdown*. Breakdown is essentially a

threshold process. No changes in the state of the medium are noticeable for some time while the electric field across a discharge gap is gradually increased. Suddenly, at a certain value of the field, instruments detect a current and even a flash can be observed.

Discharges are non-equilibrium processes that occur in initially non-ionized matter exposed to a strong electric field. Understanding the basic mechanisms of an electric discharge is a challenging problem involving ideas from non-equilibrium thermodynamics, atomic physics, electromagnetism and pattern formation. Discharges can assume many different modes of appearance depending on the spatial and temporal characteristics of the electric field and on the ionization and charge transport properties of the medium. Phenomenologically, discharges can be classified into stationary ones, such as arc, glow or dark discharges, and transient ones, such as sparks and leaders [1]. The distinctions among the various discharge phenomena seem to vary among authors.

1.1 Townsend mechanism

In 1889 F. Paschen found empirically that the breakdown characteristic of a gap is a function of the product of the gas pressure and the gap length. Paschen's law is usually written as $V = f(pd)$, where V is the voltage across the gap, p is the pressure and d is the gap distance.

Townsend managed to explain this observation. He studied the variation of the electrical current between two

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parallel plate electrodes filled with gas. The electrodes' separation and gas pressure were typically of one centimetre and one atmosphere. A high voltage was applied between the electrodes and some initial electrons were produced by illumination of the cathode with a UV source. The current in the circuit was measured for varying voltages. First there was a proportional increase in the current as the result of photoelectrons drifting towards the anode. Then, there was a range of applied voltages in which the current was constant, as the field was strong enough to enable all the liberated electrons to reach the anode, but too weak to cause any multiplication of electrons by ionization of the gas. Finally, for higher voltages, there was exponential growth. Townsend related this increase of the current to ionization of the gas by electron collisions.

Defining N as the number of electrons at distance z from the cathode, after one mean free path for ionization λ_i , an electron produces on average one electron-ion pair. So the increase in the number of electrons that can be expected in any slab of gas of thickness dz is

$$dN = N\alpha dz, \quad (1)$$

where the inverse of the ionization mean free path $\alpha = 1/\lambda_i$ is called the ionization coefficient. Integrating equation (1), yields $N = N_0 \exp(\alpha z)$, where N_0 is the number of electrons at the cathode. As a result, the electron and positive ion populations grow exponentially with distance. This process is called an electron avalanche.

The ionization coefficient α is related to the density of the neutral particles of the gas and their effective ionization cross-sections. The calculation from first principles of this coefficient is not easy. Townsend gave a phenomenological approximation by treating ionization as a process activated by drift energy gained from the field $e\lambda|\mathcal{E}|$ (e is the absolute value of the electron charge, λ is the total scattering mean free path and $|\mathcal{E}|$ is the modulus of the local electric field \mathcal{E}), with activation energy Δ . Taken all together, this leads to a formula analogous to that of Arrhenius for thermally activated processes

$$\alpha(|\mathcal{E}|) = \frac{C}{\lambda} \exp\left(\frac{-\Delta}{e\lambda|\mathcal{E}|}\right) = \alpha_0 \exp\left(\frac{-E_0}{|\mathcal{E}|}\right), \quad (2)$$

where c is a dimensionless constant and $E_0 = \Delta/e\lambda$ has the dimensions of the electric field. Since the mean free path is inversely proportional to pressure, the ionization coefficient can also be written as

$$\alpha(|\mathcal{E}|) = Ap \exp\left(\frac{-Bp}{|\mathcal{E}|}\right). \quad (3)$$

where the constants A and B are properties of the gas.

In order to reach a self-maintaining discharge, new electrons should be generated in sufficient number by secondary processes to replace the externally imposed ionization. Townsend assumed that acceleration of the positive ions in the electric field leads, in principle, to secondary emission of electrons from the negative electrode, when they reach it, at a rate of γ electrons per incident ion. The parameter γ depends on the cathode material. According to (1), N_0 initial electrons will produce $N_0 \exp(\alpha z)\alpha dz$ at position z , so across a gap of width d there will be created $N_0 [\exp(\alpha d) - 1]$ ions. The number of electrons that those ions create at the cathode should be equal to the number of electrons supplied by the UV source, so

$$\gamma N_0 [\exp(\alpha d) - 1] = N_0. \quad (4)$$

Combining (3) and (4), Townsend was able to obtain Paschen's law [1].

1.2 Streamer mechanism

Researchers found by the early 1930s that the Townsend mechanism was not able to predict the breakdown processes at atmospheric pressure and at distances over 1 cm. In order to explain the breakdown at high values of the product pd (p pressure and d distance) the streamer mechanism was developed [2,3]. Experimentally the time to breakdown is about 10–100 ns. This time is much shorter than the time it takes for the ions to move back and create secondary electrons. The breakdown voltage is independent of the cathode material. Thus the parameter γ is no longer influential. The discharge channels are sharp and narrow. This is different to the observed Townsend discharges, which are glowing, diffuse and cold. The new concept introduced to explain these facts was the effect of space charge. A streamer discharge was considered to be a plasma channel that can propagate in a gas. The discharge propagated by ionizing the medium in front of its charged head owing to a strong field induced by the head itself.

A streamer is a sharp ionization wave that propagates into a non-ionized gas, leaving a non-equilibrium plasma behind. They have also been reported in early stages of atmospheric discharges [4] such as sparks or sprite discharges [5]. Streamers can split into branches spontaneously, but precisely how this branching is determined by the underlying physics remains an open question.

To model the initial stage of dielectric breakdown, when a gas suddenly switches from being an insulating dielectric to a conducting gas, Raether [3] characterized rare long-range photoionization events as stochastic processes enhancing ionization avalanches. The idea is that photons created by atoms excited in previous collisions initiate secondary avalanches. In this scenario, avalanches are

randomly distributed at different points near the streamer head. Some phenomenological stochastic models for dielectric breakdown have been proposed since then [6]. Branching would occur due to randomly distributed ionization avalanches.

In this paper we will use classical kinetics theory to deduce a fluid description of the phenomena (if the gas density is low, it could be argued that the fluid model is no longer valid and a statistical description in terms of molecular dynamics would be more suitable). Then we will simplify the model neglecting some processes like photoionization, attachment and recombination and considering only impact ionization. We will show some numerical simulations where characteristic features of streamer propagation can be observed like finger shape and branching. We will proceed with the analysis of anode directed planar fronts and the stability of shock fronts. The dispersion curve is obtained and finally, we will present a new formulation in terms of a shielding factor which allows us to consider curved geometries.

2. Balance equations

We consider a fluid description of a low-ionized plasma based on kinetic theory [7]. Since inelastic collisions between particles play an important role in low-temperature plasmas, significant deviations from thermal equilibrium are usually present in such media, which justifies the need for using kinetic theory. In this framework, balance equations for electrons, positive and negative ions in the gas can be derived from the Boltzmann kinetic equation.

Let us take $f_e(\mathbf{R}, \mathbf{V}, \tau)$ to be the distribution function of the free electrons in a gas, where \mathbf{R} is the position vector, \mathbf{V} is the velocity and τ is time. The distribution function is defined so that $f_e(\mathbf{R}, \mathbf{V}, \tau) d^3R d^3V$ is the number of electrons which, at the instant τ , have positions inside a volume element d^3R at \mathbf{R} and have velocities lying within the volume d^3V at \mathbf{V} . The Boltzmann equation is a first-order equation of motion for the distribution function and can be written as

$$\frac{\partial f_e}{\partial \tau} + \mathbf{V} \cdot \nabla_{\mathbf{R}} f_e + \frac{F}{m} \cdot \nabla_{\mathbf{V}} f_e = \left(\frac{\partial f_e}{\partial \tau} \right)_{\text{coll}}, \quad (5)$$

where $\nabla_{\mathbf{R}}$ is the gradient in configuration space, $\nabla_{\mathbf{V}}$ is the gradient in velocity space, F is the external force acting on the particles and $(\partial f_e / \partial \tau)_{\text{coll}}$ is the time rate of change of the distribution function due to collisions. The left-hand side in (5) is the total derivative $df_e/d\tau$ and can be interpreted as the rate of change as seen in a frame moving with the particles in the six-dimensional (\mathbf{R}, \mathbf{V}) space. Thus the Boltzmann equation simply says that $df_e/d\tau$ is zero unless there are collisions.

The balance equation for electrons is obtained by integrating equation (5) in velocity space (i.e. the lowest moment of the Boltzmann equation),

$$\int \frac{\partial f_e}{\partial \tau} d^3V + \int \mathbf{V} \cdot \nabla_{\mathbf{R}} f_e d^3V + \int \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{V}} f_e d^3V = \left(\frac{\partial f_e}{\partial \tau} \right)_{\text{coll}} d^3V. \quad (6)$$

The electron density N_e is defined as the integral of the distribution function over velocity

$$N_e(\mathbf{R}, \tau) = \int f_e(\mathbf{R}, \mathbf{V}, \tau) d^3V. \quad (7)$$

With this definition, we can write the balance equation for electrons as

$$\frac{\partial N_e}{\partial \tau} + \nabla_{\mathbf{R}} \cdot (N_e \mathbf{U}_e) = S_e, \quad (8)$$

where $\mathbf{U}_e(\mathbf{R}, \tau)$ is the average (fluid) velocity of electrons and S_e is the source term, i.e. the net creation rate of electrons per unit volume as a result of collisions. It is convenient to define the electron current density as $\mathbf{J}_e(\mathbf{R}, \tau) = N_e(\mathbf{R}, \tau) \mathbf{U}_e(\mathbf{R}, \tau)$ so that the balance equation can also be written as

$$\frac{\partial N_e}{\partial \tau} + \nabla_{\mathbf{R}} \cdot \mathbf{J}_e = S_e. \quad (9)$$

The same procedure can be followed, in principle, for positive (N_p) and negative (N_n) ion densities to give

$$\frac{\partial N_p}{\partial \tau} + \nabla_{\mathbf{R}} \cdot \mathbf{J}_p = S_p, \quad (10)$$

$$\frac{\partial N_n}{\partial \tau} + \nabla_{\mathbf{R}} \cdot \mathbf{J}_n = S_n, \quad (11)$$

where $\mathbf{J}_{p,n}$ are the current densities of positive and negative ions, respectively, and $S_{p,n}$ are source terms. Conservation of charge has to be imposed in all processes, so that the condition $S_p = S_e + S_n$ holds for the source terms.

Some physical approximations can now be introduced in order to simplify the balance equations (9), (10) and (11). The first one is that the electron current \mathbf{J}_e is approximated as the sum of drift (electric force) and diffusion terms

$$\mathbf{J}_e = -\mu_e \mathcal{E} N_e - D_e \nabla_{\mathbf{R}} N_e, \quad (12)$$

where \mathcal{E} is the total electric field (the sum of the external electric field applied to initiate the propagation of an ionization wave and the electric field created by the local point charges) and μ_e and D_e are the mobility and diffusion

coefficient of the electrons. Note that, as the initial charge density is low and there is no applied magnetic field, the magnetic effects in equation (12) are neglected. This cannot be done in cases where the medium is completely ionized and/or an external magnetic field is applied (tokamaks, plasma distributions in stars, etc.) [8]. About the diffusion coefficient, in the case of equilibrium, the kinetic theory of gases links diffusion to mobility through the Einstein relation $D_e/\mu_e = kT/e$. With respect to positive and negative ions, on time scales of interest for the case of anode-directed streamers, the ion currents can be neglected because they are more than two orders of magnitude smaller than the electron one, so we will take $\mathbf{J}_p = \mathbf{J}_n = 0$.

Consider now the processes that give rise to the source terms $S_{e,p,n}$.

- (i) The first of these processes is the creation of free electrons by impact ionization: an electron is accelerated in a strong local field, collides with a neutral molecule and ionizes it. The result is the generation of new free electrons and a positive ion. The ionization rate is given by

$$S_e^i = S_p^i = v_i N_e, \quad (13)$$

where the ion production rate v_i depends on the local electric field, the density of the neutral particles of the gas and their effective ionization cross-sections.

- (ii) The second possible process, which appears in gases like oxygen, is attachment: when an electron collides with a neutral gas atom or molecule, it may become attached, forming a negative ion. This process depends on the energy of the electron and the nature of the gas [9]. The attachment rate can be written as

$$S_n^a = -S_e^a = v_a N_e, \quad (14)$$

where v_a is the attachment rate coefficient. Note that the creation of negative ions due to these processes reduces the number of free electrons, so S_e^a is negative.

- (iii) There are also two possible kinds of recombination processes: a free electron with a positive ion and a negative ion with a positive ion. The recombination rate is

$$S_e^{ep} = S_p^{ep} = -v_{ep} N_e N_p, \quad (15)$$

for electron–positive ion recombination, and

$$S_p^{np} = S_n^{np} = -v_{np} N_n N_p, \quad (16)$$

for positive ion–negative ion recombination, v_{ep} and v_{np} being the recombination coefficients respectively.

- (iv) Finally, we can include photoionization: a photon created by recombination or a scattering process can interact with a neutral atom or molecule, giving rise to a free electron and a positive ion. Models for the creation rate of electron–positive ion pairs due to photoionization are non-local [7]. This rate will be here denoted by

$$S_e^{\text{ph}} = S_p^{\text{ph}} = S^{\text{ph}}. \quad (17)$$

Liu and Pasko [10] have studied the effects of photoionization on the evolution of streamers in sprites in air. Sprites are massive but weak luminous flashes that appear above a thunderstorm and coincide with cloud-to-ground or intracloud lightning strokes. In Liu and Pasko's work, optical emissions from N_2 and N_2^+ molecules can ionize O_2 molecules. The photoionization rate is written as the following non-local source term:

$$S_{\text{ph}}(\mathbf{r}) = \int \frac{\varphi(|\mathbf{r} - \mathbf{r}_1|)}{4|\mathbf{r} - \mathbf{r}_1|^2} d^3 r_1, \quad (18)$$

where the kernel $\phi(r)$ is given by

$$\phi(r) = \frac{p_q}{p + p_q} \xi v_* N_e g(r). \quad (19)$$

In this expression, p_q is the quenching pressure of the single states of N_2 , p is the gas pressure, ξ is the average photoionization efficiency in the interval of radiation frequencies relevant to the problem, v_* is the effective excitation coefficient for the N_2 state transitions emitting the ionization radiation, and

$$g(r) = \frac{\exp(-\chi_{\min} p O_2 r) - \exp(-\chi_{\max} p O_2 r)}{r \ln(\chi_{\max}/\chi_{\min})}, \quad (20)$$

in which χ_{\min} and χ_{\max} are, respectively, the minimum and maximum absorption cross-sections of O_2 in the relevant radiation frequency interval.

Taking into account expression (12) for the current density, and equations (13), (14), (15), (16) and (17) for the source terms, we obtain a deterministic model for the evolution of the streamer discharge,

$$\frac{\partial N_e}{\partial \tau} = \nabla_{\mathbf{R}} \cdot \left(\mu_e \mathcal{E} N_e + D_e \nabla_{\mathbf{R}} N_e \right) + v_i N_e - v_a N_e - v_{ep} N_e N_p + S^{\text{ph}}, \quad (21)$$

$$\frac{\partial N_p}{\partial \tau} = v_i N_e - v_{ep} N_e N_p - v_{np} N_n N_p + S^{\text{ph}}, \quad (22)$$

$$\frac{\partial N_n}{\partial \tau} = v_a N_e - v_{np} N_n N_p. \quad (23)$$

In order for the model to be complete, it is necessary to give expressions for the source coefficients v , the electron mobility μ_e , the diffusion coefficient D_e and the photoionization source term S^{ph} .

Finally, we have to impose equations for the evolution of the electric field \mathcal{E} . This evolution of the electric field is given by Gauss law,

$$\nabla_{\mathbf{R}} \cdot \mathcal{E} = \frac{e}{\epsilon_0} (N_p - N_n - N_e), \quad (24)$$

where e is the absolute value of the electron charge, ϵ_0 is the permittivity of the gas, and we are assuming that the absolute value of the charge of positive and negative ions is e . Note that the coupling between the space charges and the electric field in the model makes the problem nonlinear.

The model given by (18), (21), (22) and (23), together with (24), has been studied numerically in the literature [10]. There are other works where the electrical current due to ions is taken into account although not photoionization [11]. In systems where there is initial background ionization, the mobility of positive ions should also be considered. Semiconductors provide one such system and the same kind of model has been applied to them [12].

3. The minimal streamer model

For simulating the dynamical streamer development of streamers out of a macroscopic initial ionization seed, in a non-attaching gas like argon or nitrogen, we can slightly simplify the model [13]. Attachment and recombination processes can be neglected ($v_a = v_{ep} = v_{np} = 0$) as these coefficients are much smaller than the ionization coefficient v_i in experimental data for non-attaching gases. As a consequence, the negative ion density N_n can be considered to be constant. With these considerations, the balance equations turn out as

$$\frac{\partial N_e}{\partial \tau} = \nabla_{\mathbf{R}} \cdot (\mu_e \mathcal{E} N_e + D_e \nabla_{\mathbf{R}} N_e) + v_i N_e + S^{\text{ph}}, \quad (25)$$

$$\frac{\partial N_p}{\partial \tau} = v_i N_e + S^{\text{ph}}. \quad (26)$$

For the ionization coefficient v_i we take the phenomenological approximation (2) given by Townsend and explained in section 1.1. This leads to a formula

$$v_i = \mu_e |\mathcal{E}| \alpha_0 \exp\left(\frac{-\mathcal{E}_0}{|\mathcal{E}|}\right), \quad (27)$$

where μ_e is the electron mobility, α_0 is the inverse of ionization length and \mathcal{E}_0 is the characteristic impact ionization electric field. Note also that $\mu_e |\mathcal{E}|$ is the drift velocity of electrons. It is usually considered that photoionization is not relevant to non-attaching gases so we can take $S^{\text{ph}} = 0$, although this is a point which deserves further investigation.

The Townsend approximation provides physical scales and intrinsic parameters of the model as long as only impact ionization is present in the gas. It is then convenient to reduce the equations to dimensionless form. The natural units are given by the ionization length $R_0 = \alpha_0^{-1}$, the characteristic impact ionization field E_0 and the electron mobility μ_e , which lead to the velocity scale $U_0 = \mu_e E_0$ and the time scale $\tau_0 = R_0/U_0$. The values for these quantities for nitrogen at normal conditions are $\alpha_0^{-1} \approx 2.3 \mu\text{m}$, $E_0 \approx 200 \text{ kV m}^{-1}$, and $\mu_e \approx 380 \text{ cm}^2 \text{ V}^{-1} \text{ s}^{-1}$. We introduce the dimensionless variables $\mathbf{r} = \mathbf{R}/R_0$, $t = \tau/\tau_0$, the dimensionless field $\mathbf{E} = \mathcal{E}/E_0$, the dimensionless electron and positive ion particle densities $\rho_e = N_e/N_0$ and $\rho_p = N_p/N_0$ with $N_0 = \epsilon_0 \mathcal{E}_0/(e R_0)$, and the dimensionless diffusion constant $D = D_e/(R_0 U_0)$.

In terms of dimensionless variables, the model equations become

$$\frac{\partial \rho_e}{\partial t} = \nabla \cdot \mathbf{j} + \rho_e f(|\mathbf{E}|), \quad (28)$$

$$\frac{\partial \rho_p}{\partial t} = \rho_e f(|\mathbf{E}|), \quad (29)$$

$$\rho_p - \rho_e = \nabla \cdot \mathbf{E}, \quad (30)$$

$$\mathbf{j} = \rho_e \mathbf{E} + D \nabla \rho_e, \quad (31)$$

$$f(|\mathbf{E}|) = |\mathbf{E}| \exp(-1/|\mathbf{E}|), \quad (32)$$

where $\nabla = \nabla_r$ and \mathbf{j} is the dimensionless electron current density. The set of equations (28)–(32) is known as the *minimal streamer model* for a non-attaching gas. In the rest of this paper we will restrict ourselves to this model. Note that we get a highly nonlinear set of partial differential equations. For this reason, even this simplified model has many interesting features for both mathematicians and physicists. Some of these features appear in the numerical simulations.

4. Simulations of the minimal model

4.1 The negative and positive fronts

We will describe a typical experimental situation to investigate the breakdown. Let us consider two separated parallel plates. The space between the plates is filled with a

non-attaching gas such as nitrogen. A stationary electric potential difference is applied to the plates. The positive charged plate is called the anode and the negative one is the cathode. An avalanche begins with a small number of seed electrons that appear accidentally (for example due to cosmic rays), or can be created irradiating the cathode or the gas with UV light [1]. A heated resistor can also be used to initiate the avalanche.

We can simulate this experiment with the minimal model based on equations (28)–(32). This is what has been done in figure 1. Cylindrical symmetry is assumed. The cathode is located at $z = 0$ and the anode is at $z = 2000$, where z is the vertical axis (we are using dimensionless coordinates). The radial coordinate extends, on the horizontal axis, from the origin at the centre of the cathode, up to $r = 2000$. The initial ionization seed is modelled by a narrow Gaussian distribution centred at $r = 0$. The diffusion is taken as $D = 0.1$. All the details about the numerics and specific parameters are discussed in [14].

The results in figure 1 correspond to a given instant of time in the evolution of the streamer. We have plotted the electron density, the positive ion density and the electric potential in the spatial region of interest. Two regions can be observed. The interior of the streamer has a non-trivial finger-like shape, as we can see in the figure. It is an ionized region that is quasi-neutral, where values of the electron density and the positive charge density are almost equal. Thus the inner region is equipotential. In the figure, the value of the potential in this region is about 110 (in dimensionless units). The outer region is filled with the non-ionized gas.

Due to impact ionization, the initial seed starts growing. If that were the only mechanism, the space charge neutrality would remain unchanged. However, the electrons have mobility in our model, so they start moving in the direction opposite to the electric field towards the anode. As the electrons drift while the ions stay put (at this fast time scale we are neglecting the ion mobility in the model as explained above), charge separation occurs. It tends to suppress the electric field inside the ionization region. When the charge separation grows in time, screening of the electric field is complete in the ionized region so the ionization process stops there. Thus it can be seen in the figure that a front is moving towards the anode leaving a quasi-neutral plasma behind. We will refer to it as the negative front. There is a very narrow region within which most of the ionization process is taking place. In this narrow layer there is a non-zero net charge density and consequently a very large electric field. Note in the figure how the equipotential lines get closer around $r = 0$ and $z = 330$, for example.

As all the physics is taking place in a very thin layer, accurate simulations are rather demanding. These features are strongly reminiscent of what occurs in combustion

fronts [15] and viscous fingering [16]. In combustion fronts, the finger region is occupied by the combustion products. This region is separated from the unburned region by a front in which chemical reactions and heat and mass transport occur. The finger solution of the combustion problem [17] is analogous to a solution of a Saffman–Taylor problem originally formulated to describe the displacement of one fluid by another having a smaller viscosity, in a porous medium or in a Hele–Shaw configuration [18].

In the above discussion we have not considered the role played by diffusion in the pictures. Without diffusion, the negative front propagates with a velocity that is at least the drift velocity of the electrons in the local electric field. To explain the movement of the streamer in the same direction as the electric field, however we need to consider diffusion explicitly. The front propagating towards the cathode will be called the positive front [13]. Its space charge separation is, as before, due to the motion of electrons when the ion–electron pair is created by an ionization event. But this time, the electrons move opposite to the diffusion current, and the propagation of the positive front is only possible if electron diffusion compensates the drift current. In the limit $D \rightarrow 0$, for a positive front to move, the electron density gradient must be extremely steep. So whereas for negative fronts, diffusion is a small correction when $D \ll 1$, in the case of positive fronts this limit is rather singular.

4.2 The branching of negative fronts

In figure 2 we can see more simulations of the model. In this case the parameters are the same as in figure 1, except that the background electric field is twice as large and the initial seed of charge is placed at the cathode, so only the evolution of the negative front towards the anode can be seen. We have plotted a few electron density level curves at two different instants of time. At $t = 365$, the electron density has the characteristic finger shape. The higher value of the electron density is found at the tip of the streamer. We observe that at time $t = 450$, the streamer has developed instabilities at the tip and these instabilities have grown out into separate fingers. Because of the imposed cylindrical geometry, the further evolution after branching ceases to be physical.

The parameters of the simulation are essentially the same as in the earlier simulations of [11]; in this reference, Vitello *et al.* had 25 kV applied over a gap of 5 mm. These parameters correspond to a dimensionless background field of 0.25 and branching was not observed. However, in figure 2 the electric field has been increased to 0.5 and the phenomena of branching appear [19]. There have been some discussions of the possibility of this branching being a numerical artefact [20]. Nevertheless such branching

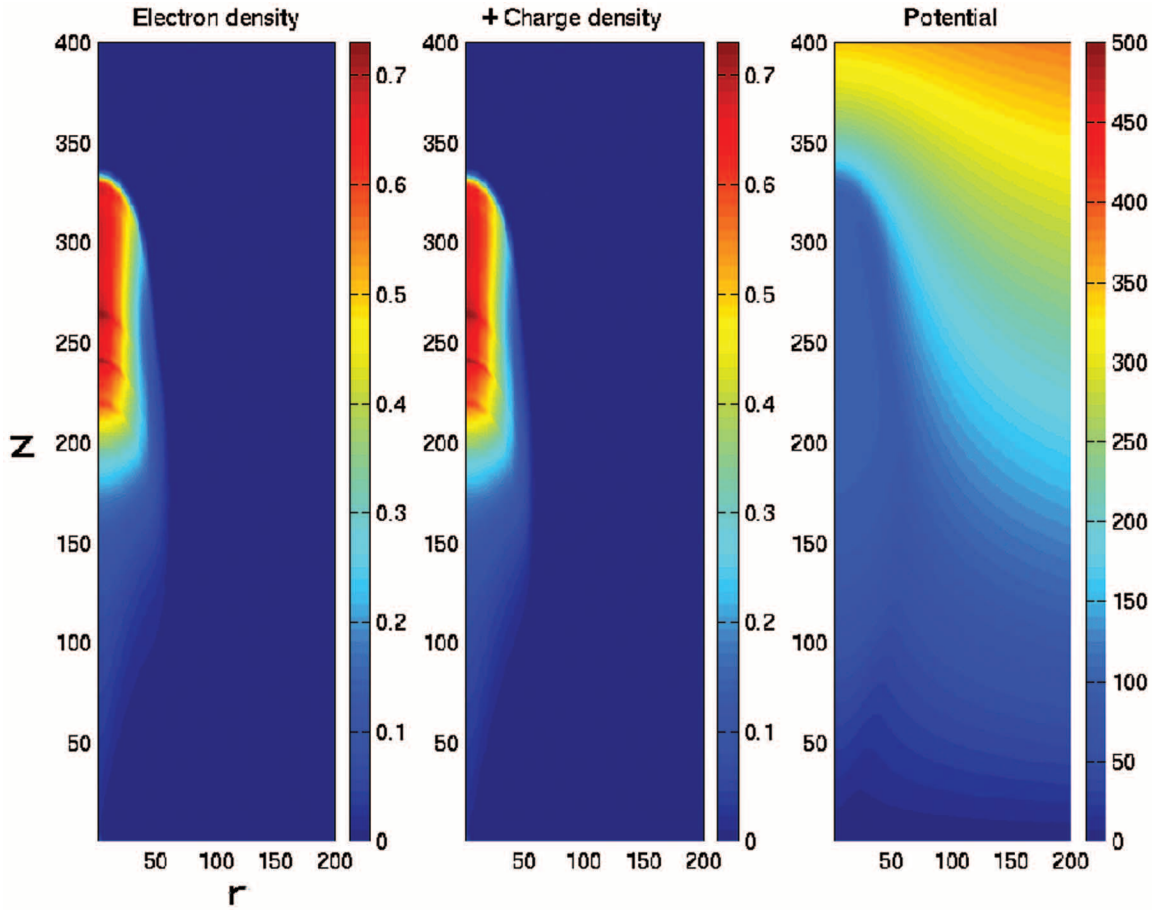


Figure 1. Electron density ρ_e , positive charge density ρ_p and electric potential. A planar cathode is located at $z = 0$ and a planar anode is at $z = 2000$, z being the vertical axis. The radial coordinate extends, on the horizontal axis, from the origin up to $r = 2000$ to avoid lateral boundary effects on the field configuration. The picture corresponds to $t = 650$ (note the use of dimensionless quantities).

phenomena have been observed in other simulations with improved accuracy and different boundary and initial conditions [10,21]. Simulations show that branching is not due to the proximity of the anode, since in a system with twice the electrode separation, the anode at $z = 4000$ and with twice the potential difference—so with the same background field—the streamer branches in about the same way after about the same time and travel distance. Moreover the phenomenon is not specific to the particular initial condition. One observes that during the temporal evolution prior to branching, both the curvature and the thickness of the ionization front decrease. So the width of the front becomes much smaller than its radius of curvature.

It has been proposed the streamers can branch spontaneously due to a Laplacian interfacial instability [19]. The conducting body has an approximately constant electric

potential, while in the non-ionized region due to the absence of space charges, the potential satisfies a homogeneous Poisson equation. The boundary between the two regions moves approximately with the drift velocity. In this scenario, branching is similar to that happening in viscous fingering phenomena [16]. The limit of ideally conducting streamers in an electric field that becomes uniform far ahead of the front was studied by Lozansky and Firsov [22].

In the following sections, we will see some analytical evidence that the streamer branching is due to an intrinsic instability of the equations of the model. Even the simplest, fully deterministic model without photoionization that we are considering can exhibit this branching. Thus branching does not occur due to randomly distributed ionization avalanches created by photo-ionization events as in other models [6].

5. The negative planar front analysis

5.1 Uniformly translating fronts

From section 4 we can argue that the relevant physics of streamer propagation operates in a very narrow space close to the fronts. Some numerical evidence of branching has been presented. We now investigate the streamer dynamics by analytical methods. The idea is to find a uniformly translating front and to investigate how a transverse perturbation of this front would develop.

We consider planar symmetry. We can then assume that charge varies only in the z direction. So the electric field can be written as $\mathbf{E} = E\mathbf{e}_z$ where $|E|$ is its modulus. A front moving uniformly with velocity v would be seen as stationary in a frame moving with the same velocity. In this comoving frame, electron and ion densities will not depend explicitly on time, so $\partial_t \rho_e = \partial_t \rho_p = 0$. We change our laboratory reference frame to a comoving frame with coordinates $(x, y, \xi = z - vt)$. Equations (28)–(32) for a uniformly translating front read

$$v\partial_\xi \rho_e + \partial_\xi(\rho_e E) + D\partial_\xi^2 \rho_e + \rho_e f(|E|) = 0, \quad (33)$$

$$v\partial_\xi \rho_p + \rho_e f(|E|) = 0, \quad (34)$$

$$\partial_\xi E - \rho_p + \rho_e = 0. \quad (35)$$

We need to determine the boundary conditions. The field is completely screened in the ionized region and is approximately constant in space and time far ahead of the front. Hence, we take E as

$$E = \begin{cases} 0 & (\xi \rightarrow -\infty) \\ E_\infty & (\xi \rightarrow +\infty) \end{cases}, \quad (36)$$

where E_∞ is a constant. These boundary conditions imply that a time independent amount of charge is travelling within the front and no currents flow far behind the front in the ionized regime.

For a non-vanishing far field E_∞ , there is a continuous family of uniformly moving front solutions [13,23]. The non-ionized region is unstable as the gas molecules tend to be ionized by the Townsend mechanism, creating a stable state. Thus the front propagates into an unstable state leaving behind a stable ionized region [24]. For $E_\infty < 0$ which is the case for the negative or anode directed front, there is a solution of the equations (33)–(36) for any value of the front velocity satisfying $v \geq |E_\infty|$. We point the interested reader to [24].

In practice, not all these uniformly propagating solutions are observed, but only a specific one selected by the initial conditions of the discharge. Let us assume that initially the electron density strictly vanishes beyond a certain point ξ_0 . Then the electron density will vanish for all times $t > 0$ in a

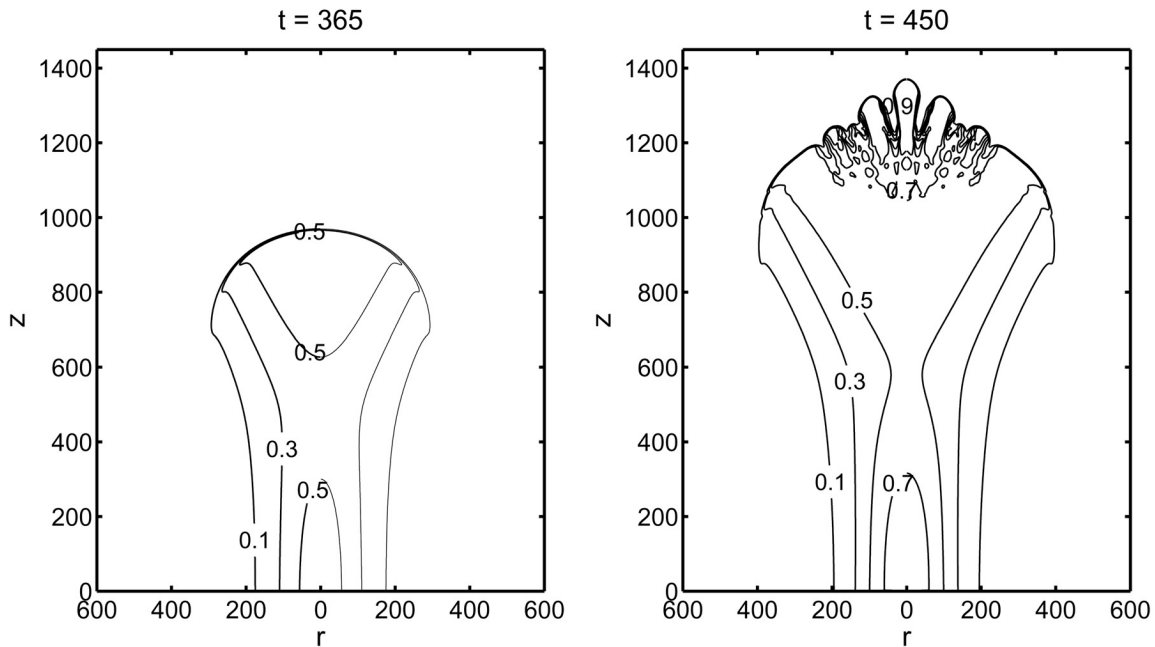


Figure 2. Spontaneous branching of anode directed streamers in a strong homogeneous background field. The planar cathode is located at $z = 0$ and the planar anode is at $z = 2000$. The radial coordinate extends from the origin up to $r = 2000$. The lines denote levels of equal electron density ρ_e with increments of 0.2 as indicated by the labels. The left picture corresponds to $t = 365$ and the right one to $t = 450$. The tip splitting of the streamer can be observed at the right picture.

coordinate system moving with velocity $v = |E_\infty|$, and an ionization front propagating with the electron drift velocity $|E_\infty|$ develops [13,24]. We will consider this case.

In figure 3 we show the solution of equations (33)–(35) fulfilling the conditions given by equations (36) and the initial conditions discussed above. We have chosen the far field $E_\infty = -1$ and the diffusion coefficient $D = 0.1$. The profiles for the electron and ion densities are plotted along with the electric field in the comoving frame. The densities decay asymptotically for large ζ as $\exp(-\lambda\zeta)$ with $\lambda \geq 0$. Around $\zeta = -10$ a non-neutral region can be observed. In this region, the electric screening of the far field starts taking place due to space charge difference $\rho_p - \rho_e$.

5.2 Shock fronts

We will further simplify the analysis by taking the limit $D = 0$ in the streamer equations. From previous discussion at the end of section 4.1, this limit is smooth for negative fronts (though it is not the case for positive fronts). It reduces the order of the equations and makes it possible to integrate them explicitly. If we take $D = 0$ in equations (33)–(35), after some straightforward manipulations, we obtain

$$-v\partial_\zeta E + \rho_e E = 0. \quad (37)$$

This equation is a consequence of charge conservation as we can see by writing $\partial_t q + \nabla \cdot \mathbf{j}_{\text{tot}} = 0$, with the total charge defined as $q = \rho_p - \rho_e$. In our model each ionizing collision produces the same number of negative and positive charges, so we have $\nabla \cdot \mathbf{j}_{\text{tot}} = 0$. The total current is given by $\mathbf{j}_{\text{tot}} = \partial_t \mathbf{E} + \rho_e \mathbf{E}$, and for a planar front with constant and time independent field $\mathbf{E} = E_\infty \mathbf{e}_z$ in the non-ionized region where $\rho_e = 0$, the total current $\mathbf{j}_{\text{tot}} = j_{\text{tot}}(t) \mathbf{e}_z$ vanishes.

We can solve equation (37) for the electric field, and then obtain the electron and ion densities analytically. The solution is a shock front [14]. In figure 4 the electron density for a shock front is plotted. The shock has a discontinuity at $\zeta = 0$.

5.3 Transversal instabilities: corrugation of the shock front

The planar shock front may be unstable with respect to periodic perturbations on the surface of discontinuity which then form ‘ripples’ or ‘corrugations’. In this case, we are interested in obtaining the dispersion relation to find which mode will grow faster and eventually determine the characteristic shape of the streamer.

Let the planar shock front that propagates into the z direction receive a small perturbation with an arbitrary dependence on the transverse coordinates x and y . Within linear perturbation theory, the perturbation can be decom-

posed into Fourier modes. Because of isotropy within the (x,y) plane, we can restrict the analysis to Fourier modes in the x direction and we consider linear perturbations proportional to $\exp(st + ikx)$. Therefore, to predict the evolution of an arbitrary transverse perturbation we need the growth rate s as a function of the wave number k . A perturbation of the densities and electric field will yield a perturbation of the position of the ionization shock front. Hence, we introduce the variable $\zeta = \xi - \epsilon \exp(ikx + st)$ and the ansatz

$$\rho_e(x, \zeta, t) = \rho_{e0}(\zeta) + \epsilon \rho_{e1}(\zeta) \exp(ikx + st), \quad (38)$$

$$\rho_p(x, \zeta, t) = \rho_{p0}(\zeta) + \epsilon \rho_{p1}(\zeta) \exp(ikx + st), \quad (39)$$

$$\phi(x, \zeta, t) = \phi_0(\zeta) + \epsilon \phi_1(\zeta) \exp(ikx + st), \quad (40)$$

where ρ_{e0} , ρ_{p0} and ϕ_0 are the electron density, ion density and electric potential of the planar ionization shock front from section 5.2. Note that these planar solutions have to be shifted to the position of the perturbed front. The substitution of the ansatz (38)–(40) into equations (33)–(35) (with $D = 0$) gives to leading order in the small parameter ϵ a set of ordinary differential equations for the quantities ρ_{e1} , ρ_{p1} and ϕ_1 .

We have to give also the boundary conditions for the perturbed quantities. There are two types of boundary condition at $\zeta = 0$, one arising from the boundedness of densities to the left of the shock front at $\zeta \rightarrow 0^-$, and the other arising from the continuity of the electric field across the position $\zeta = 0$ of the shock front. The other boundary conditions at $\zeta = -\infty$ are the total charge and the electric field vanishes. Full details of the derivation of the linear perturbation equation and the boundary conditions can be found in [14].

We have to solve an eigenvalue problem. Given k , we want to find $s(k)$ such that there is a solution for the transverse perturbation equations. The curve $s(k)$ is called the dispersion curve. In general, an analytic treatment for any value of k is not possible and we have to resort to numerical calculations [25]. However, in the limits of small and large wave number, the equations simplify and we can obtain the asymptotic behaviour of the dispersion relation $s(k)$.

We start with the small k limit. The perturbation of the front in this limit, corresponds to an infinitesimal change of the electric field E_∞ alone. If we compare two uniformly translating fronts with infinitesimally different field E_∞ at identical positions, their linearized difference solves the same equations. In this case, it can be proved [25] that the dispersion relation reads

$$s = vk + O(k^2) \text{ for } k \ll f(v)/v. \quad (41)$$

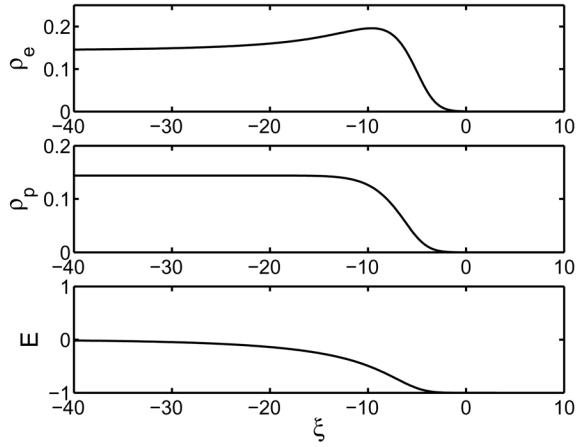


Figure 3. Electron density ρ_e , ion density ρ_p and electric field E for a negative ionization front moving with $v = |E_\infty|$ in the comoving frame. The far field $E_\infty = -1$ and $D = 0.1$.

Equation (41) has an immediate physical interpretation: $1/k$ is the largest length scale involved. It is much larger than the thickness of the screening charge layer. Therefore the charge layer can be contracted to a δ -function contribution along an interface line. Such a screening charged interface has an instability at $s = vk$.

In the opposite limit of k sufficiently large, we can also find an analytical relation for the dispersion [25]. The dispersion relation is in this case

$$s(k) = \frac{f(v)}{2} + O(k^{-1}) \text{ for } k \gg f(v)/v. \quad (42)$$

Again there is a simple physical interpretation of this growth rate. When the discontinuity propagates with velocity proportional to the local electric field $v = |E|$, a perturbation in the field $\Delta E = f(v)\zeta$ will grow with the rate $f(v)$. The averaged slope of the field for $\zeta > 0$ and $\zeta < 0$ is then $\partial_\zeta E = f(v)/2$ and this slope is precisely the growth rate in equation (42).

The linear perturbations of planar negative ionization fronts then satisfy

$$s(k) = \begin{cases} |E_\infty|k, & k \ll f(|E_\infty|)/|E_\infty|, \\ f(|E_\infty|)/2, & k \gg f(|E_\infty|)/|E_\infty|. \end{cases}$$

The planar front becomes unstable with a linear growth rate $|E_\infty|k$ for small k to a saturation value $f(|E_\infty|)/2$. We can argue that as the radius of curvature increases, the planar approximation for the tip is reasonable and allows a qualitative understanding of the branching phenomena. The results (43) show that all the modes with large enough wave number k (small wavelength perturbations) would

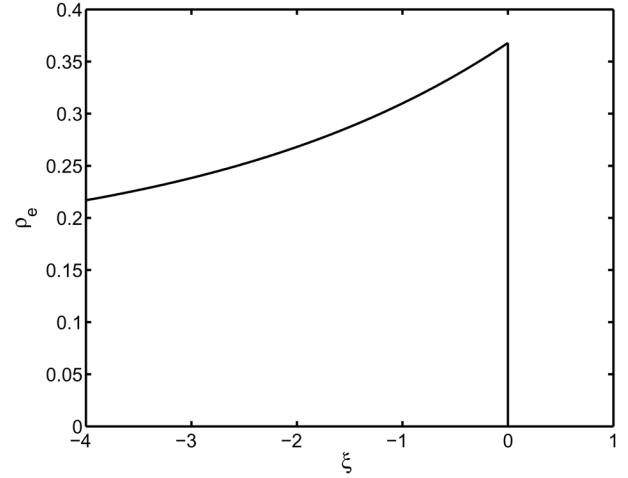


Figure 4. Electron density ρ_e for a shock front moving with $v = |E_\infty|$ in the comoving frame. The far field $E_\infty = -1$ and $D = 0$.

grow at the same rate. We expect from the physics of the problem that a particular mode would be selected. Consequently, besides the ionization characteristics length, other characteristic lengths should come into play in order to address this problem.

6. Curved negative fronts and electric shielding

The behaviour of the dispersion curve for large k for the planar case shows a saturation constant value. Modes of short wavelength will grow, becoming unstable. In order to pick up a particular one, we might consider diffusion. However, electric screening provides other characteristic lengths. In the case of curved geometries, this screening might select the mode. In the planar case it is not enough, as we have seen in the previous section, but for a non-uniform external field this issue has to be investigated (it is well known that the bigger curvature is, the stronger the electric field). In this section we will reformulate the problem of moving fronts introducing the shielding factor, following the procedure of [26]. The equation describing the evolution of the shielding factor makes easier the study of curved ionization fronts.

Let us consider again the minimal model given by equations (28)–(32) with $D = 0$. Diffusion effects are neglected in order to address the effects of curvature specifically. These equations can be reduced to a simpler form in order to give some analytic results on the evolution of the ionization fronts in planar and curved geometries. We recall that the magnetic field effects are considered to be negligible in the evolution of the ionization wave, as long as the drift velocity of the electrons is much smaller than the

light velocity. In cases in which the evolution of the streamer has some symmetry (planar, cylindrical or spherical), the electric field will have the same symmetry. If the electric field direction does not change during the discharge, we can write

$$\mathbf{E}(r, t) = \mathbf{E}_0(\mathbf{r})u(\mathbf{r}, t), \quad (44)$$

where \mathbf{E}_0 is the initial electric field. On the other hand, as the magnetic field vanishes, and the electric current is approximated as $\rho_e \mathbf{E}$, a consequence of Ampère's law is

$$\frac{\partial \mathbf{E}}{\partial t} + \rho_e \mathbf{E} = 0. \quad (45)$$

Substituting expression (44) into (45) yields

$$u(r, t) = \exp\left(-\int_0^t d\tau \rho_e(\mathbf{r}, \tau)\right). \quad (46)$$

Equation (44) reveals clearly the role played by the function $u(\mathbf{r}, t)$ as a factor modulating the electric field $\mathbf{E}(\mathbf{r}, t)$ at any time. For this reason, u can be termed a *shielding factor* and determines a screening length that depends on time. This is a kind of Debye's length which moves with the front and leaves neutral plasma behind it [26].

In terms of $u(\mathbf{r}, t)$ and the initial condition $\mathbf{E}_0(\mathbf{r})$ for the electric field, the charge densities are

$$\rho_e(\mathbf{r}, t) = -\frac{1}{u(\mathbf{r}, t)} \frac{\partial u(\mathbf{r}, t)}{\partial t}, \quad (47)$$

$$\rho_p(\mathbf{r}, t) = -\frac{1}{u(\mathbf{r}, t)} \frac{\partial u(\mathbf{r}, t)}{\partial t} + \nabla \cdot (\mathbf{E}_0(\mathbf{r})u(\mathbf{r}, t)). \quad (48)$$

These expressions are obtained substituting (44) into the minimal model equations. So the shielding factor completely determines the physics of the ionization wave in any particular problem with $D = 0$.

The above procedure reduces the problem of evolution of charged particle densities and electric field in the gas to a simpler one: to find equations and conditions for the shielding factor $u(\mathbf{r}, t)$ from equations and conditions for the quantities \mathbf{E} , ρ_e and ρ_p .

Inserting relations (44), (46)–(48) into the minimal model equations, we find that the evolution of the shielding factor is given by

$$\frac{1}{u} \frac{\partial u}{\partial t} = \nabla \cdot (\mathbf{E}_0 u) - \rho_{p0}(\mathbf{r}) - \int_{|\mathbf{E}_0|u}^{|\mathbf{E}_0|} \exp\left(\frac{-1}{s}\right) ds, \quad (49)$$

$$u(\mathbf{r}, 0) = u_0(\mathbf{r}) = 1, \quad (50)$$

in which boundary conditions should be imposed for particular physical situations. Note that we have written the complete minimal model (with $D = 0$) in one single equation for the shielding factor u . All the physics in the minimal model is contained in the evolution equations (49) and (50). Expression (49) is a Burgers-type equation with an integral term. Burgers equation is a one-dimensional, nonlinear partial differential equation similar in structure to the Navier–Stokes equation for the hydrodynamic velocity field. There are well-known methods to deal with this equation.

The shielding factor was introduced in [26] to study negative corona discharges. In this reference, it was found that, depending on initial conditions for the charge distribution, shocks or spreading fronts develop. The amplitude of these solutions decreases in time and the propagation follows a power law. Moreover diffusion phenomena due to geometrical effects appear.

7. Summary and outlook

In this paper, we have presented a fully deterministic model to investigate the initial stage of electric breakdown. The model is obtained from a Boltzmann kinetic equation taking the first moments. We discuss the principal processes such as ionization, attachment, recombination and photoionization which cause a material in a strong electric field to change its insulating properties. Then we study a reduced model which is suitable for non-attaching gases such as nitrogen, without considering photoionization. We show some numerical simulations based on this model and the evidence that an anode directed front can branch spontaneously due to a Laplacian instability. Then the stability study of a planar front and the discussion of the dispersion curve for a transversal perturbation is carried out in the absence of diffusion. Finally the formulation in terms of the shielding factor is outlined in order to study non-planar geometries.

Some questions remain to be answered. From the dispersion curve an instability will grow for a sufficiently short wavelength. We expect that a regularization mechanism should come into play. The regularization mechanism that selects a particular mode could be the electric screening due to curvature [25]. Another possibility is diffusion phenomena, which we did not consider in the case of the shock front.

The physics of low temperature plasmas is an area where many fundamental questions still remain unanswered, and where experiments have been ahead of theory. It is a difficult area due to the fact that there are neutral particles interacting with charged ones (for high temperature plasmas it is assumed that the matter is fully ionized). We agree with the words of Nobel laureate A.J. Leggett when

he said that if there is something in the conventional wisdom that you do not understand, worry away at it for as long as it takes and do not be deterred by the assurances of your fellow physicists that these questions are well understood.

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