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QUANTUM BROWNIAN MOTION IN A PERIODIC POTENTIAL: THE PATH INTEGRAL FOR A SUPER-OHMIC BATH

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The problem of a Brownian particle in a super-Ohmic bath under the effect of a periodic tilted potential is studied. The quantum dissipative dynamics is formulated in terms of the influence functional and the path integrals are calculated explicitly and analytically.

Keywords: Quantum Brownian motion in a periodic potential; the path integral for a super-ohmic bath.

1. Introduction

Brownian motion of classical particle described by a Langevin equation can be studied by a path integral formulation [1]. The path integral formalism allows some analytical insight using the method of steepest descent for calculating the leading contributions to transport properties in a very broad range of different conditions and types of thermal baths. An example of the method at work has been provided by McClintock *et al.* in the study of a ratchet driven by quasi-monochromatic noise [2].

The quantum version of the problem can be traced back to the work of Feynman and Vernon [3]. They have described the bath as an infinite set of harmonic oscillators, that in quantum mechanics represents a set of free bosonic particles and introduced a influence functional to study the dynamics. Afterwards, Caldeira and Leggett [4] have set a particular model where the coupling between the system and the bath is linear and the spectral density of the bath is Ohmic. The Hamiltonian of their model, when treated classically, is equivalent after eliminating the bath degrees of freedom to a Langevin equation. By canonically quantising the Hamiltonian, it is possible to study the motion of the Brownian particle in the quantum regime.

In this paper we will use the Caldeira–Leggett model as the starting point to study the motion of a Brownian particle in a tilted periodic potential. We will obtain the path integral formulation in the case of a super-Ohmic bath. The simpler case of an Ohmic bath has been considered previously by some authors and we will use the idea of the duality between the periodic potential and a tight binding model for the time-dependent density [5, 6].

First we will introduce the Hamiltonian of the model and characterize the super-Ohmic bath via its spectral density distribution. Then we proceed with the study of the quantum dynamics using the formalism of the influence functional. For the super-Ohmic bath, the path integrals are solved explicitly and analytically. We end with some remarks about future work and applications.

2. The Model

We will consider the Hamiltonian of a particle described by the canonical variables $\{X, P\}$ coupled linearly to an infinite set of harmonic oscillators given by $\{x_i, p_i\}$, like in the Caldeira–Leggett model. We will write it as $H = H_0 + H_I$, with

$$H_0 = \frac{P^2}{2M} + V(X), \tag{1}$$

and

$$H_I = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 \left(x_i^2 - \frac{c_i X}{m_i \omega_i^2} \right)^2. \tag{2}$$

The tilted periodic potential will be written in the most general form as

$$V(X) = -FX - \sum_{l=1}^{\infty} V_l \cos(2\pi l X / a_0 - \phi_l). \tag{3}$$

The choice of the amplitudes V_l and the phase shifts ϕ_l allows to model any periodic potential of a_0 period.

The bath is characterized by its spectral density

$$J(\omega) = \frac{\pi}{2} \sum_i \frac{c_i^2}{m_i \omega_i} \delta(\omega - \omega_i), \tag{4}$$

where m_i, ω_i and c_i are the masses, the frequencies and the coupling strengths constants of each oscillators. In this work, we will consider specifically a spectral density of the form

$$J(\omega) = \eta \omega^3 e^{-\frac{\omega}{\theta}}. \tag{5}$$

The parameter θ introduces a frequency cut-off. We have then a super-Ohmic bath [7] which has been studied in the context of the polaron problem.

3. The Influence Functional

We will use the influence functional formalism [1] to study the dynamics generated by the Hamiltonian of our system. The information on the dynamics of the Brownian particle is contained in the reduced density matrix

$$\widehat{\rho}(t) = \mathrm{Tr}_B \widehat{\rho}_{\mathrm{tot}}(t), \tag{6}$$

traced over the bath coordinates. Thus, the evolution of the average position of the particle can be obtained by the diagonal elements of the reduced density matrix.

Under the assumption that the initial total density matrix factorizes, the time evolution of the density matrix is in the coordinate representation

$$\langle X|\widehat{\rho}(t)|X'\rangle = \int dX_0 \int dX_0' \langle X_0|\widehat{\rho}(0)|X_0'\rangle \; K(X, X', t; X_0, X_0', 0), \tag{7}$$

where the kernel function K is given by the double path integral

$$K(X, X', t; X_0, X_0', 0) = \int_{X_0}^{X} DX \int_{X_0'}^{X'} DX' \mathcal{F}(X, X') \exp(i[S(X) - S(X')]/\hbar). \tag{8}$$

In (8) the action term $S(X)$ reads

$$S(X) = \int_0^t \left(\frac{M\dot{X}^2}{2} - V(X) \right) d\tau. \tag{9}$$

The $\mathcal{F}(X, X')$ is the Feynman–Vernon functional which can be written in term of the influence phase as

$$\mathcal{F}(X, X') = \exp i\Phi(X, X') \tag{10}$$

and the influence phase times the imaginary unit reads

$$\begin{aligned} i\Phi(X, X') = &-\frac{1}{\hbar} \int_0^t d\tau \int_0^\tau ds[X(\tau) - X(s)][X(s)\alpha(\tau - s) - X'(s)\alpha^*(\tau - s)] \\ &-(i/\hbar)M(\Delta\omega)^2 \int_0^t d\tau[X^2(\tau) - X'^2(\tau)]. \end{aligned} \tag{11}$$

The function $\alpha(t)$ and the renormalization term $\Delta\omega$ are related to the environment density of states $J(\omega)$ by

$$\alpha(t) = \frac{1}{\pi} \int_0^\infty d\omega J(\omega) \left[\coth(\beta\hbar\omega/2) \cos(\omega t) - i \sin(\omega t)\right], \tag{12}$$

and

$$\frac{M(\Delta\omega)^2}{2} = \frac{1}{\pi} \int_0^\infty \frac{J(\omega)}{\omega} \, d\omega. \tag{13}$$

In order to make some progress in the evaluation of the path integrals we need to expand the periodic part of potential. The periodic part of (3) is written as in [6]

$$V_p(X) = \sum_{m=\pm1,\pm2,\dots} \frac{\Delta_m}{2} e^{-2\pi i m X/a_0}, \tag{14}$$

where it is defined

$$\Delta_m = V_m e^{i\phi_m} \quad \text{for } m>0, \quad \Delta_{-m} = \Delta_m^*. \tag{15}$$

Now, we expand the exponential

$$\exp\left(\frac{i}{\hbar}\int_0^t V_p d\tau\right),$$

and use the identity

$$\frac{1}{n!}\left(\int_0^t f(\tau)d\tau\right)^n = \int_0^t dt_n \dots \int_0^{t_2} dt_1 \prod_{j=1}^n f(t_j),$$

to get

$$\begin{aligned}&\exp\left(\frac{i}{\hbar}\int_0^t V_p d\tau\right)\\ &\quad = \sum_{n=0}^{\infty}\sum_{\{m_j\}}\prod_{j=1}^n\left(\frac{i\Delta_{m_j}}{2\hbar}\right)\int_0^t dt_n \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \exp\left(-\frac{i}{\hbar}\int_0^t \rho(\tau)X(\tau)d\tau\right),\end{aligned} \tag{16}$$

where $\sum_{\{m_j\}} = \sum_{m_1=\pm1,\pm2,\dots} \cdots \sum_{m_n=\pm1,\pm2,\dots}$ means the sum on configurations of n *charges* of value of an integer number, and it has been defined the *charge density*

$$\rho(t) = \frac{2\pi\hbar}{a_0}\sum_{j=1}^n m_j \delta(t-t_j). \tag{17}$$

In order to end later dealing with Gaussian integrals, it is convenient to make the change to center of mass and relative coordinates

$$\begin{aligned} Q &= (X+X')/2,\\ q &= X - X', \end{aligned} \tag{18}$$

so the influence phase term turns out

$$\begin{aligned} i\Phi(Q,q) = &-\frac{1}{\hbar}\int_0^t d\tau \int_0^{\tau} ds[q(\tau)\alpha_R(\tau-s)q(s) + 2iq(\tau)\alpha_I(\tau-s)Q(s)]\\ &-(i/\hbar)M(\Delta\omega)^2\int_0^t d\tau q(\tau)Q(\tau), \end{aligned} \tag{19}$$

with α_R and α_I the real and imaginary part of (12).

The probability density which is necessary to obtain the mobility is the diagonal part of (7). In terms of the coordinates Q and q yields

$$P(Q,t) = \sum_{n,n'=0}^{\infty} \sum_{\{m_j,m'_{j'}\}} \prod_{j=1}^{n} \left(\frac{i\Delta_{m_j}}{2\hbar}\right) \prod_{j'=1}^{n'} \left(\frac{-i\Delta^*_{m'_{j'}}}{2\hbar}\right) \int_0^t dt_n \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1$$
$$\times \int_0^{t'_{2'}} dt'_{1'} \cdots \int_0^t dt'_{n'} \int dQ_0 \int dq_0$$
$$\times \left\langle Q_0 + \frac{q_0}{2} | \widehat{\rho}(0) | Q_0 - \frac{q_0}{2} \right\rangle \ K(\rho, \rho'; Q, Q_0, q_0), \tag{20}$$

with $K(\rho,\rho';Q,Q_0,q_0)$ the corresponding kernel.

4. The Expressions for the Super-Ohmic Case

Let us calculate K for the super-Ohmic case with an spectral density given by (5). For the influence phase (19) we have to calculate

$$M(\Delta\omega)^2 = \frac{2\eta}{\pi}\int_0^{\infty} w^2 e^{-\frac{\omega}{\theta}} d\omega = \lim_{t\to 0} \frac{-2\eta}{\pi}\int_0^{\infty} \frac{d^2(\cos\omega t)}{dt^2} e^{-\frac{\omega}{\theta}} d\omega$$
$$= \frac{2\eta}{\pi}\frac{d^2}{dt^2}\left(\frac{\theta^{-1}}{t^2+\theta^{-2}}\right)_{t=0}, \tag{21}$$

and

$$\alpha_I(t) = -\frac{\eta}{\pi}\int_0^{\infty} \omega^3 e^{-\frac{\omega}{\theta}} \sin\omega t d\omega = \frac{\eta}{\pi}\int_0^{\infty} \frac{d^3(\cos\omega t)}{dt^3} e^{-\frac{\omega}{\theta}} d\omega$$
$$= \frac{\eta}{\pi}\frac{d^3}{dt^3}\left(\frac{\theta^{-1}}{t^2+\theta^{-2}}\right). \tag{22}$$

So that,

$$\int_0^t d\tau \int_0^{\tau} ds 2q(\tau)\alpha_I(\tau-s)Q(s) = \int_0^t d\tau \int_{\tau}^t ds 2Q(\tau)\alpha_I(s-\tau)q(s)$$
$$= \frac{2\eta}{\pi}\int_0^t d\tau \int_0^{t-\tau} ds Q(\tau)\frac{d^3}{ds^3}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right) q(s+\tau)$$
$$= \frac{2\eta}{\pi}\int_0^t d\tau Q(\tau)\left\{ q(s+\tau)\frac{d^2}{ds^2}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right)\Bigg|_0^{t-\tau}\right.$$
$$- \frac{dq(s+\tau)}{ds}\frac{d}{ds}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right)\Bigg|_0^{t-\tau} + \frac{d^2q(s+\tau)}{ds^2}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right)\Bigg|_0^{t-\tau}$$
$$\left. - \int_0^{t-\tau} ds \frac{d^3q(s+\tau)}{ds^3}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right)\right\}. \tag{23}$$

Now, as in [5] we assume that the system ends at $X(t) = X'(t) = 2Q(t)$, so $q(t) = 0$. Thus the first term of the sum in (23) and the renormalization term (21) drop out of the influence phase functional, and we are left with

$$\begin{aligned} i\Phi(Q,q) = &-\frac{1}{\hbar}\int_0^t d\tau \int_0^\tau ds\ q(\tau)\alpha_R(\tau-s)q(s) \\ &+\frac{2\eta i}{\pi}\int_0^t d\tau Q(\tau)\left\{\frac{dq(s+\tau)}{ds}\ \frac{d}{ds}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right)\Bigg|_0^{t-\tau}\right. \\ &\left.-\frac{d^2q(s+\tau)}{ds^2}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right)\Bigg|_0^{t-\tau} + \int_0^{t-\tau} ds\frac{d^3q(s+\tau)}{ds^3}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right)\right\}. \end{aligned} \tag{24}$$

Note that the derivative of an even function at the origin is equal to zero and then we will impose the condition $\dot{q}(t) = 0$. Then, the first term of the imaginary part of (24) also drops out. In order to simplify the calculations, we will take the limit $\theta \to \infty$. Using the fact

$$\lim_{\theta\to\infty}\frac{1}{\pi}\left(\frac{\theta^{-1}}{s^2+\theta^{-2}}\right) = \delta(s), \tag{25}$$

and the properties of the delta function, the influence phase functional (24) simplifies tod

$$i\Phi(Q,q) = -\frac{1}{\hbar}S_R - \frac{i}{\hbar}\eta\ Q\ddot{q}\Big|_0^t + \frac{i}{\hbar}\eta\int_0^t d\tau Q(\tau)\dddot{q}(\tau), \tag{26}$$

where

$$S_R = \int_0^t d\tau\int_0^\tau ds q(\tau)\alpha_R(\tau-s)q(s), \tag{27}$$

and

$$\alpha_R(t) = \frac{\eta}{\pi}\int_0^\infty \omega^3 e^{-\frac{\omega}{\theta}}\coth(\beta\hbar\omega/2)\cos\omega t d\omega. \tag{28}$$

Finally the kernel K in (20) for the super-Ohmic case reads

$$\begin{aligned} K = \int_{Q_0}^{Q} DQ\int_{q_0}^{0} Dq\ \exp&\left\{-\frac{S_R}{\hbar} + \frac{i}{\hbar}[MQ\dot{q} - \eta Q\ddot{q}]\Big|_0^t\right. \\ &\left.+\frac{i}{\hbar}\int_0^t d\tau[-MQ\ddot{q} + \eta Q\dddot{q} + Fq - Q(\rho-\rho') - q(\rho+\rho')/2]\right\}. \end{aligned} \tag{29}$$

5. Calculating the Path Integrals

We have to calculate the double path integrals of the kernel expression (29). The path integral will be discretized as

$$DQDq = \lim_{N\to\infty}\left(\frac{\eta}{2\pi\hbar\epsilon^2}\right)^{N+1} dQ_1\cdots dQ_N dq_1\cdots dq_N, \tag{30}$$

being $\epsilon = t/N$ and the boundaries of the path integral equal to $Q_0 = Q(0)$, $q_0 = q(0)$, and $Q_{N+1} = Q(t)$, $q_{N+1} = q(t)$, respectively. With this discretization yields

$$K = \left(\frac{\eta}{2\pi\hbar\epsilon^2}\right)^{N+1} \int_{-\infty}^{+\infty} dQ_1 \cdots \int_{-\infty}^{+\infty} dq_N \exp\left\{ \frac{i}{\hbar}[MQ\dot{q} - \eta Q\ddot{q}]\Big|_0^t + \sum_{k=0}^{N} \frac{i\epsilon}{\hbar}[-MQ_k\ddot{q}_k + \eta Q_k \dddot{q}_k + Fq_k - Q_k(\rho_k - \rho_k') - q_k(\rho_k + \rho_k')/2] - \frac{S_R(q_k)}{\hbar}\right\}. \quad (31)$$

We carry out the integration for the Q_k's following that

$$K = \left(\frac{\eta}{2\pi\hbar\epsilon^2}\right)^{N+1} \int_{-\infty}^{+\infty} dq_1 \cdots \int_{-\infty}^{+\infty} dq_N (2\pi)^N \delta\left(\frac{\epsilon}{\hbar}[-M\ddot{q}_k + \eta \dddot{q}_k - (\rho_k - \rho_k')]\right) \times \exp\left\{\frac{i}{\hbar}[MQ\dot{q} - \eta Q\ddot{q}]\Big|_0^t + \sum_{k=0}^{N}\frac{i\epsilon}{\hbar}[Fq_k - q_k(\rho_k + \rho_k')/2] - \frac{S_R(q_k)}{\hbar}\right\}. \quad (32)$$

The derivatives of the path can be evaluated using the difference formulae $\dot{q}_k = (q_{k+1} - q_k)/\epsilon$, $\ddot{q}_k = (q_{k+1} - 2q_k + q_{k-1})/\epsilon^2$ and $\dddot{q}_k = (q_{k+2} - 3q_{k+1} + 3q_k - q_{k-1})/\epsilon^3$, so the delta function turns into

$$\delta\left(\frac{1}{\hbar\eta\epsilon^2}[q_{k+2} - 3q_{k+1} + 3q_k - q_{k-1} - \epsilon M(q_{k+1} - 2q_k + q_{k-1})/\eta - \epsilon^3(\rho_k - \rho_k')/\eta]\right). \quad (33)$$

We make a change of variables by

$$l_k = q_{k+2} - 3q_{k+1} + 3q_k - q_{k-1} - \epsilon M(q_{k+1} - 2q_k + q_{k-1})/\eta, \quad (34)$$

and define the vectors $\mathbf{q}^T = (q_0, \ldots, q_{N+1})$ and $\mathbf{l}^T = (l_0, \ldots, l_{N+1})$, so the transformation can be written in matrix form as

$$\mathbf{l} = \mathbf{A}\mathbf{q}, \quad (35)$$

$$\mathbf{A} = \begin{bmatrix} \left(3 + \frac{2\epsilon M}{\eta}\right) & \left(-3 - \frac{\epsilon M}{\eta}\right) & 1 & & & & \\ \left(-1 - \frac{\epsilon M}{\eta}\right) & \left(3 + \frac{2\epsilon M}{\eta}\right) & \left(-3 - \frac{\epsilon M}{\eta}\right) & 1 & & & \\ & \left(-1 - \frac{\epsilon M}{\eta}\right) & \left(3 + \frac{2\epsilon M}{\eta}\right) & \left(-3 - \frac{\epsilon M}{\eta}\right) & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & \left(-1 - \frac{\epsilon M}{\eta}\right) & \left(3 + \frac{2\epsilon M}{\eta}\right) \end{bmatrix}$$

The Jacobian of the transformation is just the inverse of the determinant of **A**. The matrix **A** is a special case of a pentadiagonal matrix, and its determinant can be written in terms of the recursive relation [8]

$$d_{-1} = 0, \quad d_0 = 1, \quad d_1 = \left(2\frac{\epsilon}{\gamma} + 3\right), \quad d_2 = \left(2\frac{\epsilon}{\gamma} + 3\right)^2 - \left(\frac{\epsilon}{\gamma} + 1\right)\left(\frac{\epsilon}{\gamma} + 3\right),$$

$$d_k = \left(2\frac{\epsilon}{\gamma} + 3\right) d_{k-1} - \left(\frac{\epsilon}{\gamma} + 1\right)\left(\frac{\epsilon}{\gamma} + 3\right) d_{k-2} + \left(\frac{\epsilon}{\gamma} + 1\right)^2 d_{k-3}, \quad k \geq 3, \tag{36}$$

where $\gamma = \eta/M$, so that $|A| = d_N$.

When (36) is rewritten in the form

$$\frac{d_k - 3d_{k-1} + 3d_{k-2} - d_{k-3}}{\epsilon^3} = \frac{2}{\gamma}\left(\frac{d_{k-1} - 2d_{k-2} + d_{k-3}}{\epsilon^2}\right) - \frac{1}{\gamma^2}\left(\frac{d_{k-2} - d_{k-3}}{\epsilon}\right), \tag{37}$$

it becomes evident that the determinant can be obtained solving a differential equation. If now let $d(t) = \epsilon^2 d_k$ for $t = k\epsilon$, we find that (37) in the limit $\epsilon \to 0 (N \to \infty)$ implies

$$\dddot{d}(t) = \frac{2}{\gamma}\ddot{d}(t) - \frac{1}{\gamma^2}\dot{d}(t)\,. \tag{38}$$

The initial values for $d(t)$ follow from

$$d(0) = \epsilon^2 d_0 \to 0, \tag{39}$$

$$\dot{d}(0) = \frac{\epsilon^2(d_1 - d_0)}{\epsilon} \to 0, \tag{40}$$

$$\ddot{d}(0) = \frac{\epsilon^2(d_2 - 2d_1 + d_0)}{\epsilon^2} \to 1, \tag{41}$$

so the solution is

$$d(t) = \gamma^2\left(\frac{t}{\gamma}e^{t/\gamma} - e^{t/\gamma} + 1\right). \tag{42}$$

Finally we can write (29) as

$$K = \frac{\eta}{2\pi\hbar d(t)} \exp\left\{\frac{i}{\hbar}\left(M\dot{q} - \eta\ddot{q}\right)Q\Big|_0^t + \frac{i}{\hbar}\int_0^t d\tau[Fq - (\rho + \rho')q/2] - \frac{S_R}{\hbar}\right\}, \tag{43}$$

where the path $q(\tau)$ is the solution of the differential equation obtained from (33) in the limit $\epsilon \to 0$

$$\eta\dddot{q} - M\ddot{q} - (\rho - \rho') = 0, \tag{44}$$

with the boundary conditions

$$q(0) = q_0, \qquad q(t) = 0, \qquad \dot{q}(t) = 0. \tag{45}$$

The general solution of (44) may be written as a sum of a particular and homogeneous solution $q = q_h + q_p$. The homogeneous solution which fulfils the boundary conditions (45) is then

$$q_h(\tau) = q_0\left(1 - \frac{\gamma^2(\frac{\tau}{\gamma}e^{t/\gamma} - e^{\tau/\gamma} + 1)}{d(t)}\right). \tag{46}$$

Now let us look for the particular solution. It has to satisfy Eq. (44) but with the boundary conditions (45) equal zero. We recall the expression for the nonhomogeneous term (17) and write

$$\rho - \rho' = \frac{2\pi\hbar}{a_0}\left[\sum_{j=1}^{n} m_j\delta(t - t_j) - \sum_{j'=1}^{n'} m'_{j'}\delta(t - t'_{j'})\right]. \tag{47}$$

Taking the Fourier transform of (44) and inserting it into S_R, at long times $t \to \infty$ the only configurations whose contributes are those for which

$$\int_0^t (\rho - \rho')d\tau = 0 \quad \text{or} \quad \sum_{j=1}^{n} m_j - \sum_{j'=1}^{n'} m'_{j'} = 0. \tag{48}$$

Defining as in [5]

$$h(\tau) = e^{\tau/\gamma}H(-\tau) + H(\tau), \tag{49}$$

where $H(\tau)$ is the Heaveside step function, the particular solution can be written as

$$q_p(\tau) = \frac{2\pi\hbar}{a_0}\gamma\left[\sum_{j'=1}^{n'} m'_{j'}h(\tau - t'_{j'}) - \sum_{j=1}^{n} m_j h(\tau - t_j)\right]. \tag{50}$$

This solution satisfies $\dot{q}_p(t) = 0$ and $q_p(t) = 0$ due to the condition (48). Up to exponentially small terms of order $\exp(-\min[t'_1, t_1]/\gamma)$ we have $q_p(0) = \dot{q}_p(0) = 0$.

6. Conclusion

To summarize the results obtained, the kernel for the super-Ohmic case in a tilted potential can be written as

$$K = \mathcal{N}\exp\left\{\frac{i}{\hbar}\left(M\dot{q}_{\rm op} - \eta\ddot{q}_{\rm op}\right)Q\Big|_0^t + \frac{i}{\hbar}\int_0^t d\tau\left[Fq_{\rm op} - \frac{(\rho + \rho')}{2}q_{\rm op}\right] - \frac{S_R}{\hbar}\right\}, \tag{51}$$

where

$$S_R = \frac{\eta}{\pi}\int_0^t d\tau\int_0^\tau ds q_{\rm op}(\tau)q_{\rm op}(s)\int_0^\infty \omega^3 e^{-\frac{\omega}{\theta}}\coth(\beta\hbar\omega/2)\cos\left(\omega(\tau - s)\right)d\omega,$$

$$\mathcal{N} = \frac{\eta}{2\pi\hbar d(t)}, \quad d(t) = \gamma^2\left(\frac{t}{\gamma}e^{t/\gamma} - e^{t/\gamma} + 1\right),$$

$$q_{\rm op}(\tau) = q_0 \left(1 - \frac{\gamma^2 \left(\frac{\tau}{\gamma} e^{t/\gamma} - e^{\tau/\gamma} + 1\right)}{d(t)}\right)$$

$$+ \frac{2\pi\hbar}{a_0}\gamma \left[\sum_{j'=1}^{n'} m'_{j'} h(\tau - t'_{j'}) - \sum_{j=1}^{n} m_j h(\tau - t_j)\right],$$

being $h(t)$ defined in (49) in terms of Heaviside functions. The conditions fulfilled by $q_{\rm op}$ are $q_{\rm op}(0) = q_0$, $q_{\rm op}(t) = \dot{q}_{\rm op}(t) = 0$ up to exponentially small terms of order $\exp(-\min[t'_1, t_1]/\gamma)$.

In this work, we have formulated the problem of a quantum Brownian particle in terms of the influence functional framework when the system under the influence of a bosonic bath in the super-Ohmic region and studied its dynamics in a tilted periodic potential. We have solved the path integrals and found explicit and analytical expressions for the reduced density matrix.

Future work will imply the calculation of the mobility of the particle, and some other quantities. The sub-Ohmic case, and other kinds of noisy environments will also be considered.

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