A class of non-null toroidal electromagnetic fields and its relation to the model of electromagnetic knots

2015 J. Phys. A: Math. Theor. 48 025203

(http://iopscience.iop.org/1751-8121/48/2/025203)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **48** (2015) 025203 (14pp) doi:10.1088/1751-8113/48/2/025203

A class of non-null toroidal electromagnetic fields and its relation to the model of electromagnetic knots

Manuel Arrayás and José L Trueba

Área de Electromagnetismo, Universidad Rey Juan Carlos, Camino del Molino s/n, E-28943 Fuenlabrada, Madrid, Spain

E-mail: manuel.arrayas@urjc.es and joseluis.trueba@urjc.es

Received 4 September 2014, revised 27 October 2014
Accepted for publication 30 October 2014
Published 11 December 2014

Abstract

An electromagnetic knot is an electromagnetic field in vacuum in which the magnetic lines and the electric lines coincide with the level curves of a pair of complex scalar fields ϕ and θ (see equations (A.1), (A.2)). When electromagnetism is expressed in terms of electromagnetic knots, it includes mechanisms for the topological quantization of the electromagnetic helicity, the electric charge, the electromagnetic energy inside a cavity and the magnetic flux through a superconducting ring. In the case of electromagnetic helicity, its topological quantization depends on the linking number of the field lines, both electric and magnetic. Consequently, to find solutions of the electromagnetic knot equations with nontrivial topology of the field lines has important physical consequences. We study a new class of solutions of Maxwell's equations in vacuum Arrayás and Trueba (2011 arXiv:1106.1122) obtained from complex scalar fields that can be interpreted as maps $S^3 \rightarrow S^2$, in which the topology of the field lines is that of the whole torus-knot set. Thus this class of solutions is built as electromagnetic knots at initial time. We study some properties of those fields and consider if detection based on the energy and momentum observables is possible.

Keywords: torus knots, electromagnetism, helicity, non-null fields
PACS numbers: 03.50.De, 42.25.Ds

(Some figures may appear in colour only in the online journal)

1. Introduction

There has been some interest in the understanding of knotted electromagnetic fields in vacuum [2–4], i.e. with some kind of linkage in the field lines, as well as knotted structures in fluids [5]. These configurations could be important for the study of the helicity exchange mechanism [6], for the stability of electromagnetic fields [7] which may play a role in particle theory [8–11], or even in certain asymptotic limits of string theory [12]. More recently for the covariant and non-abelian generalizations of the magnetic helicity [13] those fields could play a role. Also there are examples in nature where the magnetic fields created by planets and stars present also toroidal structure and nontrivial topology of the field lines [14]. In the laboratory we find such electromagnetic fields in toroidal geometries for plasma confinement [15] and in nuclear magnetic resonance devices [16]. There is evidence that injection or ejection of magnetic helicity from a plasma often affects its dynamics and stability [17]. For example the helicity content of magnetically confined plasmas in spheromaks [18] must be sustained somehow via helicity injection. On the other hand, too much magnetic helicity content can lead to violent disruptions like in the solar corona [19].

However important they are for the test of new theories and modeling, not many exact solutions of Maxwell equations in vacuum are known showing nontrivial topology. Known fields are based in the Hopf fibration [3, 6, 20–26]. Irvine and Bouwmeester have decomposed those known fields into vector spherical harmonics and, from this decomposition, produced new fields in which the helicity was conserved in time and that corresponded to all possible torus knots [2]. In their work the field lines changed topology and unraveled while evolving in time [27]. Recently Kedia *et al* [4] used Bateman's construction to build null electromagnetic fields in vacuum [28, 29]. They have obtained a complete class of null electromagnetic fields in which the electric and magnetic field lines are grouped into knotted and linked tori, nested one inside the other, with torus knots at the core of the foliation.

In the present work, we study a class of non-null solutions of Maxwell's equations in vacuum covering the topology of the whole torus knots set [1]. By having the topology of the torus knots set we mean that initially all the magnetic lines and all the electric lines are linked torus knots and, moreover, when time evolves we can find numerically field lines knotted. In figure 1 it can be seen as an example the evolution of an initial configuration for the electromagnetic field in which all the field lines are linked (4, 3) torus knots. It will be shown that the initial topology of the field lines determines the evolution of the field, and the value of quantities like the energy (18), the momentum (19) and the magnetic and electric helicities (11), (12).

The outline of the paper is the following. We first introduce the new solutions. although the more mathematical details are left for an appendix. Then we calculate some physical properties of the solutions, such as the Lorentz invariants, the electric and magnetic helicities, the energy and the linear momentum contained in those fields together with some specific dynamical quantities such as the Poynting vector. We discuss how they can be used for detection of such fields in the laboratory. The argument is that the energy and momentum are compatible observables, and the relation provided by (24) will be different depending on the topology of the created light as (25) shows. We proceed by studying some particular cases of constant helicities and finally end with some conclusions.

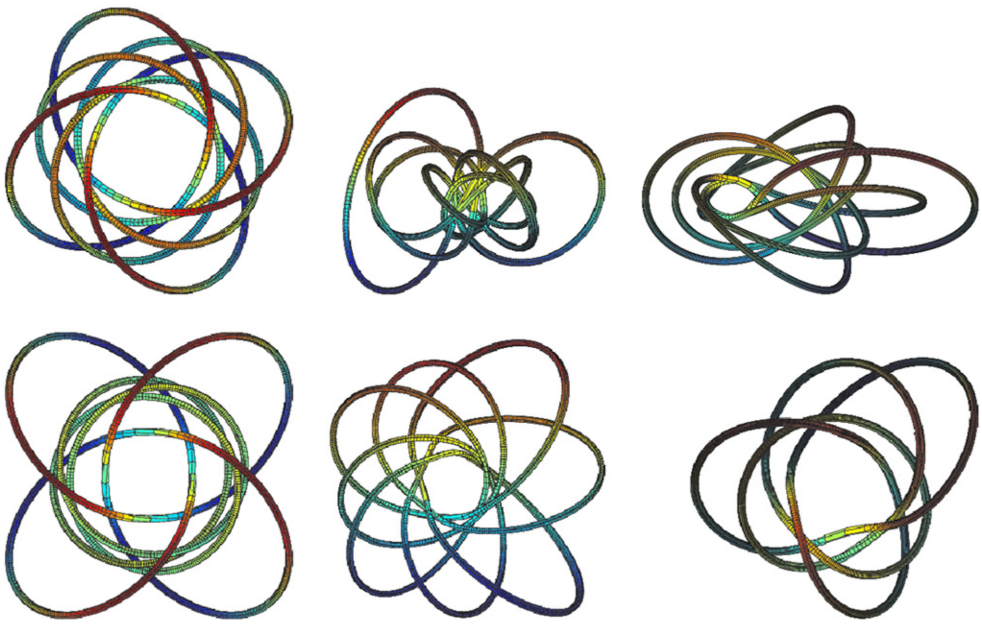

Figure 1. Time evolution of field lines for the (4, 3) torus knot. We plot two magnetic (up) and electric (down) lines for $T = 0$, $T = 0.015$ and $T = 0.02$ (left to right). Lines are obtained numerically by integrating expressions (2), (3) of the magnetic and the electric fields with respect to a curve parameter. At $T = 0$ all the field lines are linked (4, 3) torus knots and the linking number is $nm = 12$. When $T > 0$ we find (4, 3) linked torus knots.

2. Non-null electromagnetic fields with knotted field lines

It is convenient to introduce dimensionless coordinates (X, Y, Z, T), related to the physical ones (x, y, z, t) in the SI of units by $(X, Y, Z, T) = (x, y, z, ct)/L_0$, being c the speed of light. We will also use the relation $r^2/L_0^2 = (x^2 + y^2 + z^2)/L_0^2 = X^2 + Y^2 + Z^2 = R^2$, where L_0 is a constant with dimensions of length that can be considered to be the characteristic size of the knot (L_0 is related to the mean quadratic radius of the energy distribution of the electromagnetic field [3]). Thus, we have found that Maxwell equations in the vacuum

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{1}$$

are satisfied by the following fields:

$$\mathbf{B}(\mathbf{r}, t) = \frac{\sqrt{a}}{\pi L_0^2} \frac{Q\, \mathbf{H}_1 + P\, \mathbf{H}_2}{\left(A^2 + T^2\right)^3}, \tag{2}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{\sqrt{a}\, c}{\pi L_0^2} \frac{Q\, \mathbf{H}_4 - P\, \mathbf{H}_3}{\left(A^2 + T^2\right)^3}, \tag{3}$$

where the quantities A, P, Q are defined by

$$A = \frac{R^2 - T^2 + 1}{2}, \tag{4}$$

$$P = T\left(T^2 - 3A^2\right), \tag{5}$$

$$Q = A\left(A^2 - 3T^2\right), \tag{6}$$

and the vectors $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$ and $\mathbf{H}_4$ are

$$\begin{aligned}
\mathbf{H}_1 &= (-n\,XZ + m\,Y + s\,T)\,\mathbf{u}_x + (-n\,YZ - m\,X - l\,TZ)\,\mathbf{u}_y \\
&\quad + \left(n\,\frac{-1 - Z^2 + X^2 + Y^2 + T^2}{2} + l\,TY\right)\mathbf{u}_z, \\
\mathbf{H}_2 &= \left(s\,\frac{1 + X^2 - Y^2 - Z^2 - T^2}{2} - m\,TY\right)\mathbf{u}_x \\
&\quad + (s\,XY - l\,Z + m\,TX)\,\mathbf{u}_y + (s\,XZ + l\,Y + n\,T)\,\mathbf{u}_z, \\
\mathbf{H}_3 &= (-m\,XZ + n\,Y + l\,T)\,\mathbf{u}_x + (-m\,YZ - n\,X - s\,TZ)\,\mathbf{u}_y \\
&\quad + \left(m\,\frac{-1 - Z^2 + X^2 + Y^2 + T^2}{2} + s\,TY\right)\mathbf{u}_z, \\
\mathbf{H}_4 &= \left(l\,\frac{1 + X^2 - Y^2 - Z^2 - T^2}{2} - n\,TY\right)\mathbf{u}_x \\
&\quad + (l\,XY - s\,Z + n\,TX)\,\mathbf{u}_y + (l\,XZ + s\,Y + m\,T)\,\mathbf{u}_z,
\end{aligned} \tag{7}$$

where (n, m, s, l) are dimensionless positive integer numbers and a is a constant introduced so that the magnetic and electric fields have correct dimensions. In the International System of Units, that will be used in this work, a can be expressed as a pure number times the Planck constant $\hbar$ times the speed of light c times the vacuum permeability μ_0. This means that the constant a is related to the strength of the electromagnetic field. We will return later to that.

By construction, the electric field lines at a particular instant of time ($t = 0$) are linked torus knots (l, s) and the same for the magnetic field lines, which are linked torus knots of indexes (n, m). The mathematical details are left for the appendix.

3. Special properties of the torus-knotted fields

The choice of the positive integer numbers n, m, l, s determines some properties of the knotted electromagnetic fields. We start with the properties related to the Lorentz invariants of the electromagnetic fields associated to any electromagnetic field in vacuum. These invariants are $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - c^2 B^2$, that can also be obtained from the square of the Riemann–Silberstein complex vector $\mathbf{F} = \mathbf{E} + \mathrm{i}\, c\, \mathbf{B}$ [30]. From equations (2), (3)

$$\mathbf{F}(\mathbf{r}, t) = \frac{\sqrt{a}\, c}{\pi L_0^2} \frac{Q\left(\mathbf{H}_4 + \mathrm{i}\,\mathbf{H}_1\right) - P\left(\mathbf{H}_3 - \mathrm{i}\,\mathbf{H}_2\right)}{\left(A^2 + T^2\right)^3}. \tag{8}$$

For our class of knotted electromagnetic fields, with $A = (R^2 - T^2 + 1)/2$, the Lorentz invariants turns out to be

$$\mathbf{E}\cdot\mathbf{B} = \frac{ac}{\pi^2 L_0^4}\frac{1}{\left(A^2+T^2\right)^4}\Big[(ms-nl)\left(A^2+T^2\right) + (ml-ns)\left(A^2+T^2\right)(1-A)Y + 2(ls-mn)AT\left(T^2-A^2\right)\Big], \quad (9)$$

$$E^2 - c^2 B^2 = \frac{ac^2}{\pi^2 L_0^4}\frac{1}{\left(A^2+T^2\right)^4}\Big[\left(n^2-m^2\right)\left(A^2+T^2\right)\left(X^2+Y^2\right) + \left(s^2-l^2\right)\left(A^2+T^2\right)\left(Y^2+Z^2\right) + 4\left(m^2-s^2\right)A^2T^2 - \left(n^2-l^2\right)\left(A^2-T^2\right)^2\Big]. \quad (10)$$

The class of knotted electromagnetic fields has the following properties:

- *Property C1*. If the positive integer numbers n, m, l, s are equal, i.e. satisfy the condition $n = m = l = s$, then $\mathbf{E}\cdot\mathbf{B} = 0$ and $E^2 - c^2 B^2 = 0$. In terms of the Riemann–Silberstein vector $\mathbf{F}$ defined in (8), $\mathbf{F}\cdot\mathbf{F} = 0$.
- *Property C1′*. If the condition $n = m = l = s$ is not fulfilled, then $\mathbf{E}\cdot\mathbf{B} \neq 0$ and $E^2 - c^2 B^2 \neq 0$.

The time behaviour of the magnetic and electric helicities of these fields can be exactly computed. The result, in terms of the dimensionless parameter $T = ct/L_0$, is

$$h_m(t) = \frac{1}{2\mu_0 c}\int \mathrm{d}^3r\,\mathbf{A}\cdot\mathbf{B} = \frac{a}{4\mu_0}\left[(mn+ls) + (mn-ls)\frac{1-6T^2+T^4}{\left(1+T^2\right)^4}\right], \quad (11)$$

$$h_e(t) = \frac{\varepsilon_0}{2c}\int \mathrm{d}^3r\,\mathbf{C}\cdot\mathbf{E} = \frac{a}{4\mu_0}\left[(mn+ls) - (mn-ls)\frac{1-6T^2+T^4}{\left(1+T^2\right)^4}\right], \quad (12)$$

where μ_0 is the vacuum permeability, ε_0 is the vacuum permittivity, and $\mathbf{A}$ and $\mathbf{C}$ are vector potentials for the magnetic and electric fields, so that $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = \nabla\times\mathbf{C}$. The constant $a/(2\mu_0 c)$ is a unit of helicity. Knowing the initial value of the helicities and their time derivatives, the helicities can be easily obtained. The initial values are given in the appendix (see (A.5), (A.6)) using topological arguments. The time derivatives of the magnetic and electric helicities are given by

$$\frac{\mathrm{d}h_m}{\mathrm{d}t} = \frac{-1}{\mu_0 c}\int \mathrm{d}^3r\,\mathbf{E}\cdot\mathbf{B}, \quad (13)$$

$$\frac{\mathrm{d}h_e}{\mathrm{d}t} = \frac{1}{\mu_0 c}\int \mathrm{d}^3r\,\mathbf{E}\cdot\mathbf{B}. \quad (14)$$

So for this class of knotted electromagnetic fields we have

- *Property C2*. If, at $t = 0$, we have $h_m = h_e$, then for every time we have $h_m = h_e$ and both helicities are constant. That means that $mn = ls$, thus

$$\int d^3r \, \mathrm{Im}\,(\mathbf{F} \cdot \mathbf{F}) = 2\,c \int d^3r \, \mathbf{E} \cdot \mathbf{B} = 0. \tag{15}$$

As a consequence, the magnetic and electric helicities will be constant during the time evolution of the fields.

- *Property C2′*. If, at $t = 0$, we have $h_m \neq h_e$, then both helicities are not constant but, for large T, both reach the same value (due to a mechanism called exchange of helicities in [6]).

Even if the magnetic and electric helicities may change with time, the electromagnetic helicity

$$h_{em} = h_m + h_e, \tag{16}$$

is a constant of the motion for any electromagnetic field in vacuum [31]. Thus, for the electromagnetic fields given by equations (A.1), (A.2) at $t = 0$, the electromagnetic helicity satisfies

$$h_{em} = \frac{a}{2\mu_0 c}\,(nm + ls), \tag{17}$$

and this quantity is constant during time evolution since it is conserved by Maxwell's equations in vacuum.

Another important feature of these solutions is the finite value of the electromagnetic energy. The energy is related to the integer numbers n, m, l, s, that characterize the complex scalar fields ϕ and θ by

$$\mathcal{E} = \int d^3r \left(\frac{\varepsilon_0\, E^2}{2} + \frac{B^2}{2\mu_0} \right) = \frac{a}{2\mu_0 L_0}\left(n^2 + m^2 + l^2 + s^2\right). \tag{18}$$

The linear momentum of these knotted solutions can also be obtained from the Poynting vector $\mathbf{E} \times \mathbf{B}/\mu_0$ and results

$$\mathbf{p} = \int \varepsilon_0\, \mathbf{E} \times \mathbf{B} = \frac{a}{2c\mu_0 L_0}\,(ln + ms)\,\mathbf{u}_y. \tag{19}$$

A way to picture the evolution of these fields is to plot Poynting vectors at some points in space at different times. In figure 2, we see the behaviour of these electromagnetic fields for instants of time $T = -2, -1, 0, 1, 2$ (where $T = ct/L_0$). There is a focusing effect of the energy density flux at $T = 0$. In the figure we have plotted the case $(n, m) = (3, 4)$ for the initial magnetic field lines, and $(l, s) = (2, 3)$ for the initial electric field lines. The magnetic and electric fields are given by equations (2) and (3) with $n = 3$, $m = 4$, $l = 2$, $s = 3$.

There is something else about the nontrivial topological solutions of the Maxwell equations that we have found. In general, using vector identities, it is easily proved that

$$\left| \int d^3r \; \varepsilon_0 \mathbf{E} \times \mathbf{B} \right| \leqslant \int d^3r \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right). \tag{20}$$

The equality holds only when the field transport energy at the speed of light (see [32]), being the velocity at which the energy is transported defined as the time derivative of

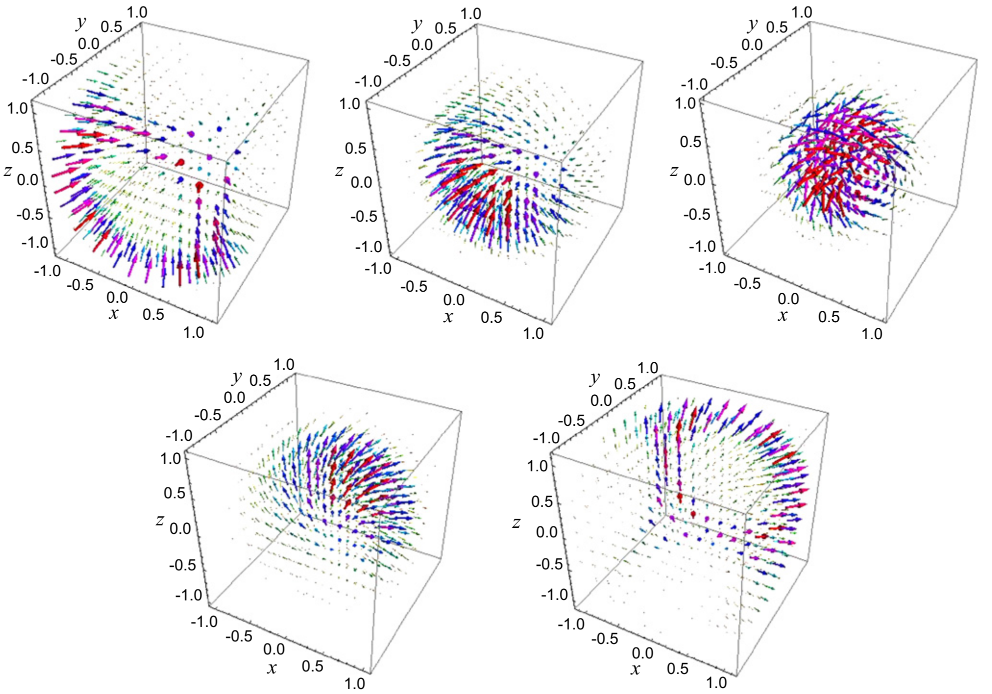

Figure 2. Time evolution of the Poynting vector $\mathbf{E} \times \mathbf{B}/\mu_0$ for the electromagnetic field given by equations (2), (3) in the case in which $m = 4$, $n = 3$, $s = 3$, $l = 2$. We plot the Poynting vectors at some points of space for $T = -2$, $T = -1$, $T = 0$, $T = 1$ and $T = 2$ (left to right, up to bottom), where $T = ct/L_0$. A focusing effect of the energy density flux is observed at $T = 0$.

$$< \mathbf{r} > = \frac{\int \mathbf{r}\, U \, \mathrm{d}^3 r}{\int U \, \mathrm{d}^3 r}, \tag{21}$$

i.e. the equality for expression (20) is obtained when

$$v_u = \left| \frac{\mathrm{d} < \mathbf{r} >}{\mathrm{d}t} \right| = c. \tag{22}$$

However, from expressions (18) and (19), we always has the strict inequality

$$p < \frac{\mathcal{E}}{c}, \tag{23}$$

being $p = |\mathbf{p}|$. Thus we can attribute a rest energy to our electromagnetic field given by the relation

$$\mathcal{E}^2 = p^2 c^2 + \mathcal{E}_0^2. \tag{24}$$

Taking into account results (18) and (19) for the energy and linear momentum of these knotted electromagnetic structures, we can find $\mathcal{E}_0$

$$\mathcal{E}_0 = \frac{a}{2\mu_0 L_0} \sqrt{\left(n^2 + m^2 + l^2 + s^2\right)^2 - (ln + ms)^2}\,. \tag{25}$$

In this way, one can see that this rest energy depends on the size of the knot, given by L_0, and on the constant a included in the definition of this kind of knotted electromagnetic structures (A.1), (A.2), which determines the energy stored in the fields. But also on the integer numbers n, m, l, s which determine the topological nature of the magnetic and electric torus knots. This quantity could be helpful in the search for knotted light in the laboratory. The energy and momentum are compatible observables, so in principle, they can be measured at the same time. The relation provided by (24) should be different depending on the topology of the created light as (25) shows.

In the model of electromagnetic knots, the constant a which appears in the amplitude definitions of the electric and magnetic fields is undetermined, but there is a natural way of relating it to other physical constants by considering the particle meaning of electromagnetic helicity [26]. The vector potential **A** can be written as a superposition of circularly polarized waves as it is done in quantum electrodynamics [33], so the electromagnetic helicity reads [25]

$$h_{em} = \hbar \int \left(\bar{a}_R(\mathbf{k}) a_R(\mathbf{k}) - \bar{a}_L(\mathbf{k}) a_L(\mathbf{k}) \right) \mathrm{d}^3\mathbf{k}, \tag{26}$$

where $a_R(\mathbf{k})$ and $a_L(\mathbf{k})$ are the Fourier components of **A** in circularly polarized waves basis, and $\bar{a}_R(\mathbf{k})$ and $\bar{a}_L(\mathbf{k})$ their complex conjugates. In QED, $a_R(\mathbf{k})$ is interpreted as a destruction operator of photonic states with energy $\hbar ck$, linear momentum $\hbar\mathbf{k}$ and spin $\hbar\mathbf{k}/k$, while the function $\bar{a}_R$ becomes the creation operator a_R^+ of such states. Analogously, $a_L(\mathbf{k})$ is interpreted as a destruction operator of photonic states with energy $\hbar ck$, linear momentum $\hbar\mathbf{k}$ and spin $-\hbar\mathbf{k}/k$, and a_L^+ is the corresponding creation operator. So that the right hand side of equation (26) becomes the helicity operator, that is the difference between the numbers of right and left handed photons. Thus

$$h_{em} = \hbar \left(N_R - N_L \right), \tag{27}$$

where N_R and N_L are the number of right and left handed photons contained in the field. This equation shows a close relation between the wave and particle aspects of the helicity.

When (27) is compared to (17), a particular value for the constant a is $a = \hbar c\mu_0$ since, in this case, one obtains the relation $N_R - N_L =$ integer for the expression of the difference between right and left handed photons included in the knotted electromagnetic field.

In macroscopic situations, as in plasma physics configurations or astrophysical contexts, the meaning of the helicity is not related to the photon content of the field at all, only to the linkage of the field lines. The constant a can take any other value to give the right units.

4. Study of the field lines of knotted electromagnetic fields with constant helicities ($l = n$ and $s = m$)

In the previous section we have seen how to construct analytically different electromagnetic fields in vacuum for which, at $t = 0$, all the magnetic lines and all the electric lines are linked torus knots in the space R^3, except a zero-measure set. For this fields, at $t = 0$ all the magnetic lines are the same torus knot, and all the electric lines are the same torus knot. But in general the kind of torus knot corresponding to the magnetic lines may be different from the kind of torus knot corresponding to the electric lines.

Next we are going to restrict ourselves to the case in which, at $t = 0$, both the magnetic and the electric lines are the same kind of torus knot. This means that, in equations (2)–(7), we take $l = n$ and $s = m$. Consequently, the fields satisfy properties $C1'$ and $C2$, so that the magnetic field is not orthogonal to the electric field but the magnetic and the electric helicities

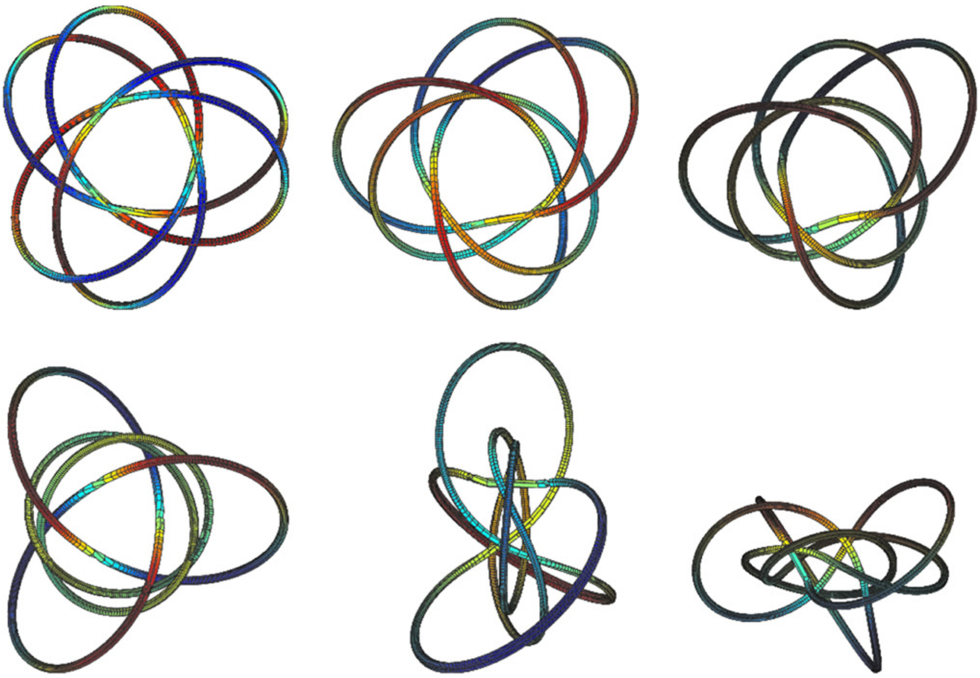

Figure 3. Time evolution of field lines for the (3, 2) torus knot or trefoil. We plot two magnetic (up) and electric (down) lines for $T = 0$, $T = 0.015$ and $T = 0.02$ (left to right). Lines are obtained numerically by integrating expressions (2), (3) of the magnetic and the electric fields with respect to a curve parameter. Numerical integration shows that at $T = 0$ all the magnetic lines (and the electric lines) are trefoils except two lines (as always occurs in torus knots): the straight line $X = Y = 0$ and the circle $X^2 + Y^2 = 1$. Moreover, the trefoils are linked to each other so that the linking number is $nm = 6$. When $T > 0$ we find numerically magnetic (and electric) lines that are linked trefoils.

are equal and constant in time. In [2], a different method to construct these kind of fields was proposed.

To see how these solutions give different torus knots, we plot in figure 3 some magnetic and electric lines for the particular case $n = 3$, $m = 2$. At $T = 0$ the lines are linked trefoils and the linking number is $H(\phi) = H(\theta) = nm = 6$. When $T > 0$ we also find a set of linked trefoils, although not all the lines are linked trefoils at $T > 0$. In figure 1 we see the behaviour of magnetic and electric lines in the case $n = 4$, $m = 3$. At $T = 0$, all the lines are linked (4, 3) torus knots and the linking number is $nm = 12$. For $T > 0$ we find lines with the same topology. The same is done in figure 4 with $n = 5$, $m = 2$ and linking number 10. In all these figures it is clear that plotted curves are linked knots. When $T = 0$, these are the only kind of curves that can be found since these lines are level curves of the scalar fields (A.3), (A.4). As T increases, numerical evidence prevent us to rule out the existence of some open field lines together with the knotted closed ones. A similar behaviour was found also numerically in [2].

In the special case in which we set solutions of the equations (3)–(7) for which $n = m = l = s$, we have a situation in which properties *C1* and *C2* hold. These particular solutions, called Hopf–Rañada electromagnetic knots [2, 3, 20, 21, 24, 25], are based on the Hopf fibration. They constitute a kind of basis for general electromagnetic fields in vacuum with nontrivial topology of the field lines.

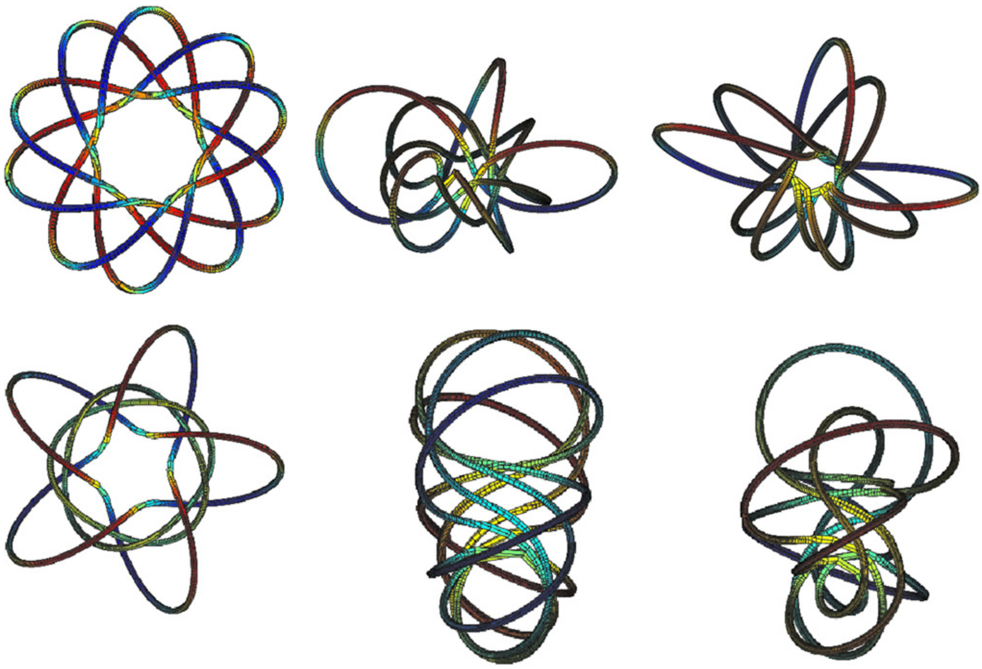

Figure 4. Time evolution of field lines for the (5, 2) torus knot. We plot two magnetic (up) and electric (down) lines for $T = 0$, $T = 0.015$ and $T = 0.02$ (left to right). Lines are obtained numerically by integrating the explicit time dependent expressions (2), (3) of the fields respect to a curve parameter. At $T = 0$ all the field lines are linked (5, 2) torus knots and the linking number is $nm = 10$. When $T > 0$ we find (5, 2) linked torus knots.

5. Conclusions

In this work we have studied a class of knotted electromagnetic fields. The solutions found are such that, at a given initial time, they satisfy that the magnetic lines and the electric lines are linked torus knots and they include as particular cases the Hopf–Rañada electromagnetic knots. Expressions (2)–(7) are the main results of this paper.

We have found some properties satisfied by these solutions, as the values of the Lorentz invariants and the magnetic and electric helicities. The Lorentz invariants associated to these solutions depend on the position and time, and the magnetic and electric helicities also depend on time. One of the intriguing properties they have is that, when time is large, the magnetic and electric helicities become equal. This is what is called the helicity exchange mechanism discovered in [6].

An energy and momentum specific relations can be associated to each one of these solutions. One might consider to detect these electromagnetic fields by measuring the energy–momentum content of the fields. That is related to the topology of the electromagnetic field lines.

We have computed numerically the field lines corresponding to cases in which the magnetic and electric helicities are constant in time. These are situations in which the initial electric lines and the initial magnetic lines are the same kind of torus knots. When time evolves, we have observed in the numerical computations of the field lines some curves with the topology of the given torus knot. Since the magnetic and electric helicities of these configurations are average measures of the linking number of the field lines and they are equal and conserved, it can be argued that some kind of nontrivial topology of these solutions can be found at any time.

Acknowledgments

We thank Wolfgang Löffler, José M Montesinos and Antonio F Rañada for discussions and encouragement. This work has been partially supported by the Spanish Ministerio de Economía y Competitividad, under the projects AYA2011-29936-C05-03 and ESP2013-48032-C5-2-R.

Appendix. Construction of a class of electromagnetic fields with knotted field lines

Torus knots are knotted curves lying on the surface of a torus. Any torus knot is defined by two coprime integer numbers (n, m) in such a way that the curve winds n times around a circle inside the torus and m times around a line through the hole in the torus.

Here is a method to find some knotted electromagnetic fields in vacuum (see [26]), i.e. exact solutions of Maxwell's equations in empty space in which the magnetic lines and/or the electric lines are knotted curves.

Let $\phi(\mathbf{r})$, $\theta(\mathbf{r})$, two complex scalar fields that depend on the position $\mathbf{r}$ in the physical space R^3. In the cases that we are going to study in this work, both scalar fields can be considered as maps $\phi, \theta : S^3 \to S^2$ after identifying the physical space R^3 with S^3 (doing so, it is assumed that both scalars have only one value at infinity) and the complex plane with S^2. These identifications can be done using stereographic projections and, as we will see later, have important consequences in the solutions of Maxwell's equations that we are going to obtain. However, a similar formalism could be applied to other possibilities in which both scalars cannot be considered as maps from S^3 to S^2. For example, this formalism could be used to construct electromagnetic fields in vacuum from the scalars considered in [34] to implement optical beams.

If ϕ (or θ) is a map from S^3 to S^2, then the preimage of any point in S^2 is a closed curve in S^3. Moreover, the linking number of a pair of curves in S^3 (that are preimages of two distinct points in S^2) is the same for all the pairs of curves (except at most for a zero-measure set, a property common to torus knots). The linking number of each pair of curves is called the Hopf invariant $H(\phi)$ of the map ϕ and it is an integer number.

We will impose that, at some initial time $t = 0$, the level curves of the complex scalar fields (ϕ, θ) coincide with the magnetic and electric lines respectively, each one of these lines being labelled by the constant value of the corresponding scalar. This can be simply done by constructing the magnetic and electric fields at $t = 0$ as

$$\mathbf{B}(\mathbf{r}, 0) = \frac{\sqrt{a}}{2\pi \mathrm{i}} \frac{\nabla\phi \times \nabla\bar{\phi}}{\left(1 + \bar{\phi}\phi\right)^2}, \tag{A.1}$$

$$\mathbf{E}(\mathbf{r}, 0) = \frac{\sqrt{a}\, c}{2\pi \mathrm{i}} \frac{\nabla\bar{\theta} \times \nabla\theta}{\left(1 + \bar{\theta}\theta\right)^2}. \tag{A.2}$$

The notation $\bar{\phi}$ means the complex conjugate of ϕ and i is the imaginary unit. Note that, using vector identities, the expressions (A.1), (A.2) are such that they satisfy $\nabla \cdot \mathbf{B} = 0$ and

$\nabla \cdot \mathbf{E} = 0$ for any scalar fields ϕ and θ, which is a necessary condition for $\mathbf{B}(\mathbf{r}, 0)$ and $\mathbf{E}(\mathbf{r}, 0)$ to be admissible initial values of an electromagnetic field in vacuum.

The construction given in equations (A.1), (A.2), along with the fact that ϕ and θ are stereographic projections of maps from S^3 to S^2, assures that all pairs of lines of the field $\mathbf{B}(\mathbf{r}, 0)$ are linked, and that the linking number is the same for all the pairs and it is given by the Hopf index $H(\phi)$ of the map ϕ. Similarly, all pairs of lines of the field $\mathbf{E}(\mathbf{r}, 0)$ are linked, the linking number of all pairs of lines is the same and it is given by the Hopf index $H(\theta)$ of the map θ. Thus, at $t = 0$, the linkage of all the magnetic and the electric lines is set by this construction.

Our starting point is the following choice of complex scalar fields

$$\phi = \frac{(X + \mathrm{i}Y)^{(n)}}{\left(Z + \mathrm{i}\left(R^2 - 1\right)/2\right)^{(m)}}, \tag{A.3}$$

$$\theta = \frac{(Y + \mathrm{i}Z)^{(l)}}{\left(X + \mathrm{i}\left(R^2 - 1\right)/2\right)^{(s)}}, \tag{A.4}$$

where n, m, l and s are positive integer numbers. These fields are related to the Seifert fibrations [35]. The main difference is that, in equations (A.3), (A.4), the notation $\eta^{(n)}$, η being a complex number, means to leave the modulus of η invariant while the phase of η is multiplied by n.

Level curves of the complex scalar field ϕ given by expression (A.3) are (n, m) linked torus knots. This means that if we choose any complex number, say $1 + \mathrm{i}$, the equation $\phi = 1 + \mathrm{i}$ gives a curve in the space R^3, and this curve is a (n, m) torus knot. If we choose any other value, say $4 - 7\mathrm{i}$, the equation $\phi = 4 - 7\mathrm{i}$ gives another (n, m) torus knot. The tangent vectors of such curves are parallel to $\nabla\, \mathrm{Re}\,(\phi) \times \nabla\, \mathrm{Im}\,(\phi)$. Moreover, both curves are linked and their linking number is, precisely, the Hopf index of ϕ, that is $H(\phi) = nm$ in this case. This occurs for any level curves of ϕ. Since level curves of ϕ coincide, through equation (A.1), with magnetic lines at $t = 0$, we can say that any pair of magnetic lines at $t = 0$ is a linked pair of (n, m) torus knots, and that the linking number is nm. The same can be said about the complex scalar field θ given by equation (A.4) and the electric field at $t = 0$ given by equation (A.2), so that any pair of electric lines at $t = 0$ is a linked pair of (l, s) torus knots and the linking number is $H(\theta) = ls$. To solve Maxwell's equations in vacuum with the initial conditions given by expressions (A.1), (A.2), Fourier analysis can be used to get the exact solutions.

The magnetic helicity h_m of these fields, which is a measure of the mean value of the linkage of the magnetic lines [36], and the electric helicity h_e, have initial values related, through equations (A.1), (A.2), to the Hopf indices of ϕ and θ, respectively. This is only true when (i) the magnetic and the electric fields have the form given by equations (A.1), (A.2), and (ii) the scalar fields ϕ and θ are stereographic projections of maps from S^3 to S^2. Consequently, the initial values of the magnetic and the electric helicities of these electromagnetic fields are given by

$$h_m(t = 0) = \frac{a}{2\mu_0} H(\phi), \tag{A.5}$$

$$h_e(t = 0) = \frac{a}{2\mu_0} H(\theta), \tag{A.6}$$

where μ_0 is the vacuum permeability, ε_0 is the vacuum permittivity, and **A** and **C** are vector potentials for the magnetic and electric fields, so that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = \nabla \times \mathbf{C}$. The constant $a/(2\mu_0 c)$ is a unit of helicity. For $t \neq 0$, equations (A.5), (A.6) are not true. This is due to the fact that the linkage of the field lines may change during time evolution (see [6] for an example, in which this kind of change was presented).

However, there is also an important case, called the Rañada–Hopf electromagnetic knot, in which $n = m = l = s = 1$, where the magnetic and electric helicities take the values given by equations (A.5), (A.6) for any time. In this case, the linkage of magnetic or electric lines is conserved in time. It has been analytically proof that all the magnetic and electric lines remain closed and linked for any value of T since there is an explicit expression for the scalar fields ϕ and θ of equations (A.1), (A.2) for any time. These time dependent expressions were firstly found in [25] and they are

$$\phi = \frac{(A\,X - T\,Z) + \mathrm{i}(A\,Y + T\,(A - 1))}{(A\,Z + T\,X) + \mathrm{i}(A\,(A - 1) - T\,Y)}, \tag{A.7}$$

$$\theta = \frac{(A\,Y + T\,(A - 1)) + \mathrm{i}(A\,Z + T\,X)}{(A\,X - T\,Z) + \mathrm{i}(A\,(A - 1) - T\,Y)}, \tag{A.8}$$

where A is given in equation (4). The time evolution of both scalars is smooth and this implies that the field lines remain closed and the topology is conserved.

In the general case with different values of the integer numbers n, m, l, s, it is not possible to find the scalar fields ϕ and θ for $T > 0$.

References

[1] Arrayás M and Trueba J L 2011 Electromagnetic torus knots arXiv:1106.1122
[2] Irvine W T M and Bouwmeester D 2008 *Nat. Phys.* **4** 716
[3] Arrayás M and Trueba J L 2010 *J. Phys. A: Math. Theor.* **43** 235401
[4] Kedia H, Bialynicki-Birula I, Peralta-Salas D and Irvine W T M 2013 *Phys. Rev. Lett.* **111** 150404
[5] Kleckner D and Irvine W T M 2013 *Nat. Phys.* **9** 253–8
[6] Arrayás M and Trueba J L 2012 *Ann. Phys.* **524** 71
[7] Rañada A F and Trueba J L 1996 *Nature* **383** 32
[8] Thomson W 1868 *Trans. R. Soc. Edinburgh* **25** 217
[9] Wheeler J A 1957 *Phys. Rev.* **97** 511
[10] Jehle H 1971 *Phys. Rev.* D **3** 306
[11] Jehle H 1972 *Phys. Rev.* D **6** 441
[12] Muñoz G and Tennant D 2009 *Phys. Lett.* B **682** 297
[13] Buniy R V and Kephart T W 2014 *Annals Phys.* **344** 179
[14] Innes D E, Inhester B, Axford W I and Wilhelm K 1997 *Nature* **386** 811
[15] Taylor J B 1974 *Phys. Rev. Lett.* **33** 1139
[16] Jackson J A, Burnett L J and Harmon F 1980 *J. Magn. Reson.* **41** 411
[17] Hsu S C and Bellan P M 2010 *IEEE Trans. Plasma Sci.* **30** 10–11
[18] Bellan P M 2000 *Spheromaks: a Practical Application of Magnetohydrodynamic Dynamos and Plasmas Self-Organization* (London: Imperial College)
[19] Golub L and Pasachoff J M 1997 *The Solar Corona* (Cambridge: Cambridge University Press)
[20] Trautman A 1977 *Int. J. Theor. Phys.* **16** 561
[21] Rañada A F 1989 *Lett. Math. Phys.* **18** 97
[22] Rañada A F 1990 *J. Phys. A: Math. Gen.* **23** L815
[23] Rañada A F 1992 *J. Phys. A: Math. Gen.* **25** 1621
[24] Rañada A F and Trueba J L 1995 *Phys. Lett.* A **202** 337
[25] Rañada A F and Trueba J L 1997 *Phys. Lett.* A **235** 25
[26] Rañada A F and Trueba J L 2001 *Modern Nonlinear Optics: Part 3* ed M W Evans (New York: Wiley) pp 197–253

[27] Irvine W T M 2010 *J. Phys. A: Math. Theor.* **43** 385203
[28] Hogan P 1984 *Proc. R. Soc.* A **396** 199
[29] Bialynicki-Birula I 2004 *J. Opt. A: Pure Appl. Opt.* **6** S181
[30] Bialynicki-Birula I 1996 The photon wave function *Coherence and quantum optics: part 12* ed J H Eberly, L Mandel and E Wolf (New York: Plenum) p 313
[31] Deser S and Teiltelboim C 1976 *Phys. Rev.* D **13** 1592
[32] Schwinger J *et al* 1998 *Classical Electrodynamics* (Cambridge: Westview)
[33] Hatfield B 1992 *Quantum Field Theory of Point Particles and Strings* (Reading, CA: Addison-Wesley)
[34] Dennis M R, King R P, Jack B, O'Holleran K and Padgett M J 2010 *Nat. Phys.* **6** 118
[35] Dufraine E 2002 *Qual. Theory Dyn. Syst.* **3** 361
[36] Moffatt H K and Ricca R L 1992 *Proc. R. Soc.* A **439** 411