Mechanism of Branching in Negative Ionization Fronts

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When a strong electric field is applied to nonconducting matter, narrow channels of plasma called streamers may form. Branchlike patterns of streamers have been observed in anode directed discharges. We explain a mechanism for branching as the result of a balance between the destabilizing effect of impact ionization and the stabilizing effect of electron diffusion on ionization fronts. The dispersion relation for transversal perturbation of a planar negative front is obtained analytically when the ratio D between the electron diffusion coefficient and the intensity of the externally imposed electric field is small. We estimate the spacing λ between streamers and deduce a scaling law $\lambda \sim D^{1/3}$.

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One of the greatest unsolved problems in the physics of electric discharges is the clarification of the mechanism of branching. When a strong electric field is applied to a nonconducting medium such as a gas, narrow channels of ionized matter called streamers may be formed. The phenomenon has been observed in a wide range of scales, from small-gap discharges of few a centimeters up to scales of kilometers such as in sprites discharges from a thunder cloud. The pattern of this branching resembles the ones observed in the propagation of cracks, dendritic growth, and viscous fingering. Those phenomena are known to be governed by deterministic equations rather than by stochastic events. In this Letter, for the first time, it is derived a quantitative prediction of the branching characteristic length based on a deterministic model. The results give an explicit dependence of branching with the electric field and the pressure of the gas as it has been observed qualitatively in experiments [1].

We use a fluid approximation to describe electric breakdown in nonattaching gases such as nitrogen. In these gases, there are indications showing that the most important source of electrons and positive ions from neutral molecules is impact ionization [2]. This is the only process that is taken into account in this work, leading to a dimensionless minimal streamer model [3,4] which reads

$$\frac{\partial N_e}{\partial \tau} = \nabla \cdot (N_e \boldsymbol{\mathcal{E}} + D_e \nabla N_e) + N_e |\boldsymbol{\mathcal{E}}| e^{-1/|\boldsymbol{\mathcal{E}}|}, \quad (1)$$

$$\frac{\partial N_p}{\partial \tau} = N_e |\boldsymbol{\mathcal{E}}| e^{-1/|\boldsymbol{\mathcal{E}}|}, \quad (2)$$

$$N_p - N_e = \nabla \cdot \boldsymbol{\mathcal{E}}, \quad (3)$$

from which the electron N_e and positive ion N_p densities can be obtained. $\boldsymbol{\mathcal{E}}$ is the dimensionless local electric field in the gas and D_e is the dimensionless electronic diffusion coefficient. The scales of these quantities for nitrogen [5] depend on the gas pressure p. The characteristic field, length, time, velocity, particle density, and diffusion coefficient are $\mathcal{E}_0 = 200$ kV/cm$(p/1$ bar$)$, $R_0 = 2.3$ μm$(p/1$ bar$)^{-1}$, $\tau_0 = 3$ ps$(p/1$ bar$)^{-1}$, $U_0 = 76 \times 10^6$ cm/s$(p/1$ bar$)^0$, $N_0 = 4.7 \times 10^{14}$ cm$^{-3}(p/1$ bar$)^2$ and $D_0 = 1.8 \times 10^4$ cm^2/s$(p/1$ bar$)^{-1}$ respectively.

Equation (1) means that the electron density varies at a point as a result of (i) the electric current contribution $\nabla \cdot (N_e \boldsymbol{\mathcal{E}})$, (ii) the electronic diffusion term $D_e \nabla^2 N_e$, and (iii) the impact ionization source term $N_e |\boldsymbol{\mathcal{E}}| e^{-1/|\mathcal{E}|}$. Equation (2) means that the positive ion density in a point of the gas only varies in time as a result of impact ionization, since the mobility of ions is much smaller than that of the electrons. Equation (3) is Poisson's law for the electric field, which is supposed to be irrotational since the magnetic effects are neglected.

In the minimal model, a spontaneous branching of negative streamers has been observed numerically [4], as it occurs in experimental situations [6]. In order to understand this branching, the dispersion relation for transversal Fourier modes of planar negative shock fronts (without diffusion) has been derived [7]. For perturbations of small wave number k, the planar shock front becomes unstable with a linear growth rate proportional to k, but all the modes with large enough wave number k seem to grow at the same rate.

In this Letter, we address the problem of the selection of a particular wave number. We will obtain a new dispersion relation depending explicitly on the electric field and the electronic diffusion coefficient. Our analysis will show that the electron density N_e may develop steep fronts of thickness $O(\sqrt{D_e})$, satisfying an equation analogous to the Fisher equation [8]. A surprising fact established during the last 30 years is that the combination of sharp interfaces with small diffusive effects may result in asymptotic limits (for $D_e \ll 1$) in which the motion of the interface is such that its points move along the normal direction with a velocity proportional to its mean curvature [9]. This analysis concerned a model known today for incorporating the Allen-Cahn equation and this kind of dynamics is termed "mean curvature flow." Many authors have contributed to provide a rigorous proof of the convergence of the Allen-Cahn model to motion by mean curvature and exported the ideas to various other systems [9–13]. Remarkably, some of these limiting models may be such that the solutions develop branchlike patterns. In this work, we exploit some of these ideas to study the motion of ionization fronts. We show that a planar front separating a (partly) ionized region from a region without charge is such that small geometrical perturbations in the charge distribution lead to a motion of the front affected by two opposed effects: electrostatic repulsion of electrons and electron diffusion. The first effect tends to destabilize the front while the second acts effectively as a mean curvature contribution to the velocity of the front, thus stabilizing it. The net result is the appearance of fingers with a characteristic thickness determined by the balance of these two opposing actions.

We consider the following situation. The space between two large planar plates, situated at $x = 0$ (cathode) and $x = d$ (anode) (x is the horizontal axis and we suppose that $d \gg 1$), is filled with a nonattaching gas like nitrogen. A stationary electric potential difference is applied to these plates, so that an electric field is produced in the gas. The initial electric field is directed from the anode to the cathode and is uniform in the space between the plates. We first study the evolution of planar negative ionization fronts towards the anode.

We concentrate on the study of the dynamics under the effect of strong external electric fields. We denote the modulus of the dimensionless electric field at a large distance from the cathode as $\mathcal{E}_\infty$ and assume that $\mathcal{E}_\infty \gg 1$. Under these circumstances, it is natural to rescale the dimensionless quantities in the minimal model as $\boldsymbol{\mathcal{E}} = \mathcal{E}_\infty \mathbf{E}$, $N_e = \mathcal{E}_\infty n_e$, $N_p = \mathcal{E}_\infty n_p$, and $\tau = t/\mathcal{E}_\infty$. For $\mathcal{E}_\infty \gg 1$, since $\exp(-1/[\mathcal{E}_\infty|\mathbf{E}|]) = 1 - O(1/[\mathcal{E}_\infty|\mathbf{E}|]) \approx 1$, the system can be approximated by

$$\frac{\partial n_e}{\partial t} - \nabla \cdot (n_e \mathbf{E} + D \nabla n_e) = n_e |\mathbf{E}|, \tag{4}$$

$$\frac{\partial n_p}{\partial t} = n_e |\mathbf{E}|, \tag{5}$$

$$\nabla \cdot \mathbf{E} = n_p - n_e, \tag{6}$$

where $D = D_e/\mathcal{E}_\infty$ is, in general, a small parameter. Our approximation will be valid in all regions where $|\mathbf{E}| = O(1)$. These are the regions of interest since by Eq. (6), the electric field is not expected to vary much in the neighborhood of the ionization front as long as n_p and n_e are bounded. We will prove below that it is the case for the solutions of (4)–(6).

In the evolution of an ionization wave along the x axis, the rescaled electric field can be written as $\mathbf{E} = E\mathbf{u}_x$ where $E < 0$, so that $|\mathbf{E}| = |E| = -E$ and $\mathbf{u}_x$ is an unitary vector in the x direction. It is very simple to compute traveling wave solutions propagating with velocity c when $D = 0$. It can be shown [14] that these solutions exist for any $c \geq 1$. We will be interested in the limit $c \to 1$ since it is well known that a compactly supported initial data (representing a seed of ionization located in some region) develops fronts traveling with this velocity. In the case $c = 1$ the solution can be obtained straightforwardly, giving (with $\xi = x - ct$ and $c = 1$)

$$E(\xi) = \begin{cases} -e^{\xi}, & \text{for } \xi < 0 \\ -1, & \text{for } \xi \geq 0 \end{cases}, \quad n_e(\xi) = \begin{cases} 1, & \text{for } \xi < 0 \\ 0, & \text{for } \xi \geq 0 \end{cases},$$
$$n_p(\xi) = \begin{cases} 1 - e^{\xi}, & \text{for } \xi < 0 \\ 0, & \text{for } \xi \geq 0 \end{cases}. \tag{7}$$

In the case $0 < D \ll 1$, it is known [15] that all initial density profiles decaying at infinity faster than $Ae^{-\lambda^* x}$, with $\lambda^* = 1/\sqrt{D}$, will develop traveling waves with velocity $c = 1 + 2\sqrt{D}$. If $D \ll 1$, the profiles for n_p and E will vary very little from the profiles with $D = 0$. On the other hand, n_e will develop a boundary layer at the front smoothing the jump from $n_e = 1$ to $n_e = 0$. Using Eq. (6), the term $\nabla \cdot (n_e \mathbf{E})$ can be written as $\mathbf{E} \cdot \nabla n_e + n_e(n_p - n_e)$. Approximating at the boundary layer $n_p = 0$, $E = -1$, and introducing $n_e(x, t) = n_e(\chi)$ with $\chi = [x - (1 + 2\sqrt{D})t]/\sqrt{D}$, Eq. (4) results in

$$-2\frac{\partial n_e}{\partial \chi} - \frac{\partial^2 n_e}{\partial \chi^2} = n_e(1 - n_e), \tag{8}$$

together with the matching conditions $n_e(-\infty) = 1$ and $n_e(+\infty) = 0$. Equation (8) is the well-known equation for the traveling waves of Fisher's equation. It appears in the context of mathematical biology [16] and is known to have solutions subject to our matching conditions. This means that we have a boundary layer of width $\sqrt{D}$ at $\xi = 0$ in which Eq. (8) gives the solution for the electron density n_e. Before this layer, we have $n_e \approx 1$, and after the layer, $n_e \approx 0$. When $D = 0$, this is the shock front of Eq. (7). Concerning the profile of n_p at the boundary layer, since Eq. (5) can be written as

$$\frac{\partial n_p(\chi)}{\partial t} = -\frac{1 + 2\sqrt{D}}{\sqrt{D}} \frac{dn_p(\chi)}{d\chi} = n_e|\mathbf{E}| \approx n_e, \tag{9}$$

at lowest order in D one obtains

$$n_p(\chi) = -\sqrt{D} \int_\chi^\infty n_e(\chi) d\chi, \tag{10}$$

so that $\partial n_p/\partial x$ is $O(1)$ at the boundary layer.

Now we make a perturbation in the transversal direction y. We introduce a new system of coordinates in the form $\bar{t} = t$, $\bar{y} = y$, $\bar{x} = x - \delta\varphi(x, y, t)$ so that, at $t = 0$, $n_e^{(0)}(\bar{x})$, $n_p^{(0)}(\bar{x})$, and $E^{(0)}(\bar{x})$ correspond to the profiles of the traveling wave computed in the previous paragraph, and δ is a small parameter compared to $\sqrt{D}$ (see Fig. 1). By doing this, we follow a strategy analogous to the one used in Rubinstein *et al.* [10] to deduce the asymptotic approximation of Allen-Cahn equation by mean curvature flow.

We introduce the perturbed electric field and electron and ion densities as

$$\mathbf{E} = E^{(0)}\mathbf{u}_x + \delta(E_x^{(1)}\mathbf{u}_x + E_y^{(1)}\mathbf{u}_y), \tag{11}$$

$$n_e = n_e^{(0)} + \delta^2 n_e^{(1)}, \tag{12}$$

$$n_p = n_p^{(0)} + \delta n_p^{(1)}, \tag{13}$$

and we will select the function $\varphi(x, y, t)$ and the $O(\delta)$ corrections $E_x^{(1)}$, $E_y^{(1)}$, $n_p^{(1)}$ in such a way that Eqs. (11)–(13) represent correct series expansions. Specifically, the correction to the traveling wave profiles for n_e will be $O(\delta^2)$.

We insert these expressions into Eqs. (4)–(6). Then we impose that $O(\delta^0)$ terms and $O(\delta^1)$ terms vanish. The solution of the equations at order $O(\delta^0)$ is the traveling wave found previously, so that $n_e^{(0)}(\bar{x}, \bar{t}) = n_e^{(0)}(\bar{x} - \bar{t})$ is given by Fisher's equation (8), $n_p^{(0)}(\bar{x} - \bar{t})$ is given by Eq. (10), and $E^{(0)}(\bar{x} - \bar{t})$ is the solution of Poisson's equation corresponding to these particle densities.

At order $O(\delta^1)$, after some manipulations, and taking into account that $\partial/\partial\bar{x} = O(D^{-1/2})$ and $\partial n_p^{(0)}/\partial\bar{x} = O(1)$ at the boundary layer, and that D is a small parameter, the equation for the perturbed ion density $n_p^{(1)}$ decouples from the equations for the perturbed electron density $n_e^{(1)}$ and the perturbed electric field $(E_x^{(1)}, E_y^{(1)})$. The system at $O(\delta^1)$ order is then given by the evolution equations

$$\frac{\partial\varphi}{\partial\bar{t}} + E_x^{(1)} - D\nabla^2_{(\bar{x},\bar{y})}\varphi - 2D\frac{\partial\varphi}{\partial\bar{x}}\frac{\partial^2 n_e^{(0)}/\partial\bar{x}^2}{\partial n_e^{(0)}/\partial\bar{x}} = 0, \tag{14}$$

$$\frac{\partial E_x^{(1)}}{\partial\bar{x}} + \frac{\partial E_y^{(1)}}{\partial\bar{y}} - \frac{\partial\varphi}{\partial\bar{x}}(n_p^{(0)} - n_e^{(0)}) = 0. \tag{15}$$

Observe that the system (14) and (15) simplifies if one assumes that φ is independent of $\bar{x}$. This is a valid assumption if φ is independent of $\bar{x}$ at $\bar{t} = 0$ and while $E_x^{(1)}$ is independent of $\bar{x}$. These conditions hold in the $O(D^{1/2})$ boundary layer around $\bar{x} = 0$ at all times, as we will discuss from Eq. (17).

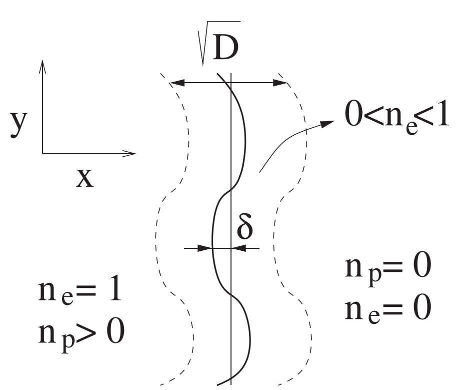

FIG. 1. A planar front is perturbed by a displacement of characteristic size δ. The boundary layer has a characteristic size $\sqrt{D}$ as explained in the text. In this boundary layer, the electron density n_e goes from 1 to 0 and the positive ion density n_p decays to 0. The front moves to the right along the positive x axis.

It is more convenient to formulate Eq. (15) in terms of the electric potential. We note that the total electric field has to be irrotational since the magnetic field is negligible. So we will assume that $\mathbf{E} = -\nabla V$, where V is an electric potential that can be written as $V(\bar{x}, \bar{y}) = V^{(0)}(\bar{x}) + \delta V^{(1)}(\bar{x}, \bar{y})$. At order $O(\delta^0)$, Poisson's equation implies that $V^{(0)}(\bar{x})$ is an electric potential associated with the electric field $E^{(0)}(\bar{x})$. At order $O(\delta^1)$, Poisson's equation implies that $V^{(1)}$ satisfies

$$\nabla^2_{(\bar{x},\bar{y})}V^{(1)}(\bar{x},\bar{y}) = \nabla^2_{(\bar{x},\bar{y})}\varphi\frac{\partial V^{(0)}(\bar{x})}{\partial\bar{x}} + 2\frac{\partial\varphi}{\partial\bar{x}}\frac{\partial^2 V^{(0)}(\bar{x})}{\partial\bar{x}^2} - n_p^{(1)}, \tag{16}$$

with the condition of decaying at $|\bar{x}| \to \infty$. Again, by noting $\partial/\partial\bar{x} = O(D^{-1/2})$, we can neglect $n_p^{(1)}$ in Eq. (16), and solve the resulting equation by taking Fourier transform in $\bar{y}$ to find the following value for the Fourier transform of $E_x^{(1)}$,

$$\hat{E}_x^{(1)}(\bar{x}, k) = -\frac{|k|\hat{\varphi}(k)}{2} \times \begin{cases} \frac{1}{1+|k|}e^{-|k|\bar{x}}, & \text{for } \bar{x} \geq 0 \\ \frac{-2|k|}{1-|k|^2}e^{\bar{x}} + \frac{1}{1-|k|}e^{|k|\bar{x}}, & \text{for } \bar{x} \leq 0 \end{cases}. \tag{17}$$

The front is at a neighborhood of $O(D^{1/2})$ width around $\bar{x} = 0$. While $|k| \ll D^{-1/2}$, the exponentials in Eq. (17) can be neglected in this region, and we can write

$$\hat{E}_x^{(1)}(0, k) = -\frac{|k|\hat{\varphi}(k)}{2(1 + |k|)}, \tag{18}$$

i.e., a field independent of $\bar{x}$.

Assuming that φ does not depend on $\bar{x}$, we can write Eq. (14) in the form

$$\frac{\partial\varphi}{\partial\bar{t}} + E_x^{(1)} - D\frac{\partial^2\varphi}{\partial\bar{y}^2} = 0. \tag{19}$$

The third term on the left hand side of Eq. (19) is the contribution of the electron diffusion to the evolution of the front and is also its linearized mean curvature. Taking the Fourier transform of Eq. (19) in $\bar{y}$, and using Eq. (18), we find

$$\frac{\partial\hat{\varphi}(k)}{\partial\bar{t}} - \frac{|k|\hat{\varphi}(k)}{2(1 + |k|)} + D|k|^2\hat{\varphi}(k) = 0. \tag{20}$$

In order to solve the ordinary differential equation (20), we substitute $\hat{\varphi}(k, \bar{t}) = e^{s\bar{t}}\hat{\phi}(k)$ into Eq. (20), which yields

$$s = \frac{|k|}{2(1 + |k|)} - D|k|^2, \tag{21}$$

and that gives the dispersion curve of the transversal perturbations of the planar negative ionization front explicitly in terms of the parameter $D = D_e/\mathcal{E}_\infty$ (see Fig. 2). Note that in the original time variable τ, the dispersion relation is (21) with the right hand side multiplied by $\mathcal{E}_\infty$.

From this result we can obtain some important consequences on the branching of streamers: (i) There exists a maximum of $s(|k|)$ that selects the wavelength of the perturbation. When D is a small parameter, this maximum is approximately located at

$$k_{\max} \approx \left(\frac{\mathcal{E}_\infty}{4D_e}\right)^{1/3}. \tag{22}$$

Notice that $k_{\max}$ is $O(D^{-1/3})$, so that $|k|\bar{x}$ can be safely approximated by zero in the boundary layer. This justifies the assumption, introduced previously in this work, that $E_x^{(1)}$, and hence φ, are independent of $\bar{x}$ at this order. The value of $k_{\max}$ corresponds to a typical spacing between fingers given by

$$\lambda_{\max} = \frac{2\pi}{k_{\max}} \approx 10\left(\frac{D_e}{\mathcal{E}_\infty}\right)^{1/3}. \tag{23}$$

This is an equation in dimensionless units. If we introduce typical scales for nitrogen [see the paragraph below Eq. (3)], then we can obtain the typical distance between two consecutive branches in a negative streamer discharge. This distance will decrease when the gas pressure or the initial electric field in the gas increases. (ii) There exists a stability threshold, i.e., a value of k delimiting the stability region, which corresponds to the nontrivial zero of the function $s(k)$,

$$k_0 \approx \sqrt{\frac{\mathcal{E}_\infty}{2D_e}}. \tag{24}$$

The wavelength λ_0 associated with this wave number is

$$\lambda_0 = \frac{2\pi}{k_0} \approx 8.9\sqrt{\frac{D_e}{\mathcal{E}_\infty}}. \tag{25}$$

Although the predictions are made for negative planar fronts, they agree with the observed fact that, for positive discharges, the number of streamers increases with the electric field and the pressure. Those effects are accounted by the expression (23).

We can now provide a qualitative mechanism of streamer branching. When the radius of a streamer becomes larger than the critical length given by Eq. (25), the streamer becomes unstable and branching develops. The electric field which should be taken into Eq. (23) in case of inhomogeneous electric discharges is the local field at the front of the streamer. In order to test the predictions, experimental evidence should be provided in the range where the approximations of large electric fields and small diffusion coefficient are valid.

To conclude, we have derived analytically the characteristic length of branching for planar negative ionization fronts. At the same time, this prediction can be considered as a test for the validity of the minimal deterministic model on which this calculation is based.

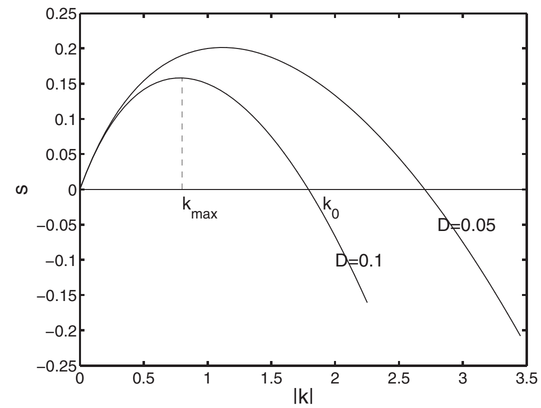

FIG. 2. The dispersion relation given by Eq. (21) for $D = 0.1$ and $D = 0.05$. For the first curve, the position of the maximum ($k_{\max}$) and the value where the curve changes sign (k_0) are shown. The physical roles of these values are discussed in the text.

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