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The quest of null electromagnetics knots from Seifert fibration

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ABSTRACT

In this work we find new null electromagnetic fields that are exact solutions of Maxwell equations in vacuum and generalize the hopfion. The hopfion is an exact solution of Maxwell equations in vacuum in which all the field lines (both electric and magnetic) are topologically equivalent to closed and linked circles, forming a mathematical structure called Hopf fibration. Here we present a generalization to include other field lines topology, such as the Seifert fibration in which the field lines form linked torus knots. Included in this generalization are fields that ergodically fill torus surfaces.

1. Introduction

The search for solutions of Maxwell equations with non-trivial topology was pioneering by Antonio F. Rañada. In a seminal paper [1] he found the first example of an electromagnetic knot, the celebrated hopfion [2]. The hopfion is an electromagnetic field in which all the field lines are topologically equivalent to linked and closed circles. The topology of the field lines implies some interesting physical consequences and some attempts to find new solutions has been carried on. There has been in the literature two main directions for trying to generalize Rañada solutions and find more complex fibrations in Maxwell equations in vacuum.

In [3], the hopfion was generalized so that the new electromagnetic fields still satisfied the same null conditions as the hopfion. The null conditions impose the Lorentz invariants of the field to be zero, i.e., $\mathbf{E}\cdot\mathbf{B} = 0$ and $E^2 - c^2B^2 = 0$. The magnetic lines of the new solutions were organized around a set of core field lines which are closed and form (n, m) linked torus knots when n and m are integers and coprimes. The hopfion is contained in this class for the particular values $n = m = 1$. However, not all the field lines at every instant of time had the topology of the Seifert fibration. It is important to note that, in these new solutions, the electromagnetic fields cannot be written as in Rañada's formulation of electromagnetic knots,

$$\mathbf{B}(\mathbf{r},t) = \frac{\sqrt{a}}{2\pi i}\frac{\nabla\phi\times\nabla\bar{\phi}}{(1+\phi\bar{\phi})^2}, \tag{1}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{\sqrt{ac}}{2\pi i}\frac{\nabla\bar{\theta}\times\nabla\theta}{(1+\theta\bar{\theta})^2}, \tag{2}$$

where $\phi(\mathbf{r},t)$ and $\theta(\mathbf{r},t)$ are complex fields, the bar over the fields denoting complex conjugation. The constant a is defined so that the magnetic and electric fields have the right dimensions (MKS units are used) and c is the speed of light in vacuum. Note that, as a consequence, the constraint imposed by the condition that all the magnetic field lines are level curves of a complex scalar field ϕ and all the electric lines are level curves of a complex scalar field θ is not guaranteed.

In [4], the generalization was done in the complex fields ϕ and θ, but the fields did not satisfy the null conditions. New solutions, covering the topology of the whole torus knots set and having the hopfion as a particular case, were found. In those solutions, all the magnetic lines and all the electric lines were initially linked (n, m) torus knots, so Seifert fibration was obtained at $t = 0$, but this structure was not maintained when $t \neq 0$.

In this work, we consider the problem of finding exact solutions of Maxwell equations in vacuum such that

1. The magnetic and electric fields have to be given by complex scalar fields ϕ and θ as in Eqs. (1)–(2).
2. The magnetic and electric fields have to satisfy the null conditions $\mathbf{E}\cdot\mathbf{B} = 0$ and $E^2 - c^2B^2 = 0$.

In our solutions, at a particular time ($t = 0$ without loss of generality) we find cases where the field lines are linked and closed (both magnetic and electric). Those cases constitute a generalization of the Hopf fibration, including a Seifert fibration. Moreover, the new family of solution also include cases in which the field lines ergodically fill torus shape regions.

2. Complex scalar fields and the null condition

Since, in [4], Seifert fibrations were found at $t = 0$, we will use a similar, though more general, choice of the complex scalar fields ϕ and

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θ,

$$\phi = \frac{(v_1 + i\,v_2)^{(n)}}{(v_3 + i\,v_4)^{(m)}}, \tag{3}$$

$$\theta = \frac{(v_2 + i\,v_3)^{(n)}}{(v_1 + i\,v_4)^{(m)}}, \tag{4}$$

where n and m are positive real numbers and the notation $f^{(n)}$, f being a complex function, means

$$f^{(n)} = \frac{f^n}{(f\bar{f})^{(n-1)/2}}. \tag{5}$$

We will take

$$v_1(\mathbf{r}) = \frac{X}{\sqrt{R^2+\delta^2}}, \tag{6}$$

$$v_2(\mathbf{r}) = \frac{Y}{\sqrt{R^2+\delta^2}}, \tag{7}$$

$$v_3(\mathbf{r}) = \frac{Z}{\sqrt{R^2+\delta^2}}, \tag{8}$$

$$v_4(\mathbf{r}) = \frac{\delta}{\sqrt{R^2+\delta^2}}, \tag{9}$$

so that $v_1^2 + v_2^2 + v_3^2 + v_4^2 = 1$. The hopfion case corresponds to the particular choice $\delta = (R^2-1)/2$, $n = m = 1$. The X, Y, Z symbols represent dimensionless Cartesian coordinates. The physical coordinates x, y, z can be obtained using the relations

$$x = L_0 X, \quad y = L_0 Y, \quad z = L_0 z, \tag{10}$$

L_0 being a constant with dimensions of length related to the mean quadratic radius of the electromagnetic energy distribution [5]. The positive dimensionless radial coordinate R is also defined, so that $r = L_0 R$, with

$$r^2 = x^2 + y^2 + z^2 = L_0^2(X^2+Y^2+Z^2) = L_0^2 R^2. \tag{11}$$

We will consider the case δ being a function of the square of the radial coordinate R, thus

$$\delta = \delta(R^2). \tag{12}$$

By using the Rañada formulation (1)–(2), the magnetic and electric fields associated to the complex scalar fields (3)–(4) are

$$\mathbf{B} = \frac{\sqrt{a}}{\pi L_0^2 (R^2+\delta^2)^2}\,[-mM\,(Y,-X,\delta) - nNZ\,(X,Y,Z) - (n-m)\delta M\,(0,0,1)], \tag{13}$$

$$\mathbf{E} = \frac{c\sqrt{a}}{\pi L_0^2 (R^2+\delta^2)^2}\,[mM\,(\delta,Z,-Y) + nNX\,(X,Y,Z) + (n-m)\delta M\,(1,0,0)]. \tag{14}$$

In these equations, we have defined

$$M = \delta - 2R^2\delta', \tag{15}$$

$$N = 1 + 2\delta\delta', \tag{16}$$

and

$$\delta' = \frac{d\,\delta}{d\,R^2}. \tag{17}$$

Now, we impose null conditions $\mathbf{E}\cdot\mathbf{B} = 0$ and $E^2 - c^2B^2 = 0$ to the initial fields (13)–(14). What we get is that both null conditions are satisfied as long as

$$-m^2M^2 + n^2\left(2MN\delta + N^2R^2\right) = 0, \tag{18}$$

where δ is a function of R^2 and M, N are given in Eqs. (15)–(16). In the next section, we examine solutions of Eq. (18).

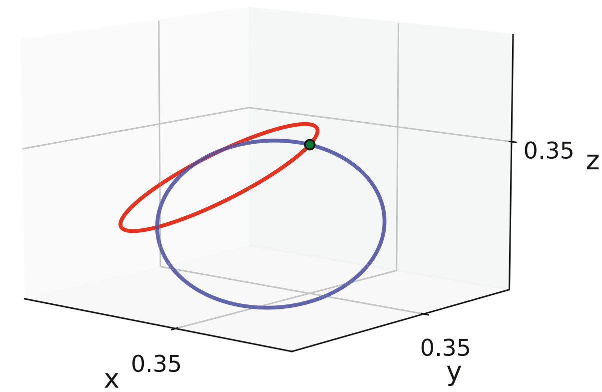

Fig. 1. Reobtaining the celebrated hopfion for $m = n$. Electric (blue) and magnetic (red) lines are perpendicular at the common point $(0.35, 0.35, 0.35)$ in dimensionless space units. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

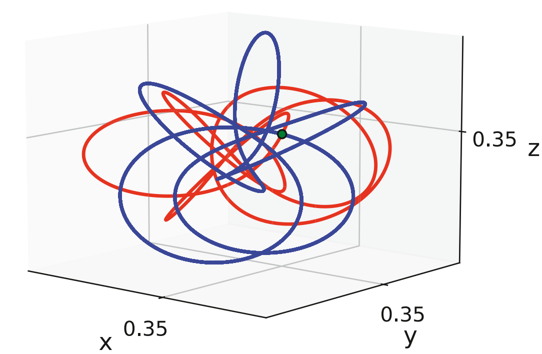

Fig. 2. The case $m = 4$ and $n = 5$. Electric (blue) and magnetic (red) lines are perpendicular at the common point $(0.35, 0.35, 0.35)$ in dimensionless space units. All the lines are closed torus knots $(4, 5)$ in this case. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

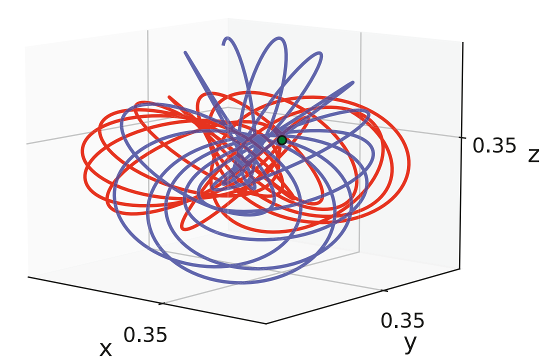

Fig. 3. The case $m = \sqrt{3}$ and $n = 2$. Electric (blue) and magnetic (red) lines are perpendicular at the common point $(0.35, 0.35, 0.35)$ in dimensionless space units. The lines are not closed but describe a torus shape region. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3. Solutions of the characteristic equation

In this section, we present a method to find solutions of the fundamental Eq. (18), which gives our fields the null character. Substituting expressions (15)–(17) into (18), we get

$$4(\delta')^2R^2 - 4\delta\delta' - \frac{\delta^2(2-\lambda) + R^2}{\delta^2 + \lambda R^2} = 0, \tag{19}$$

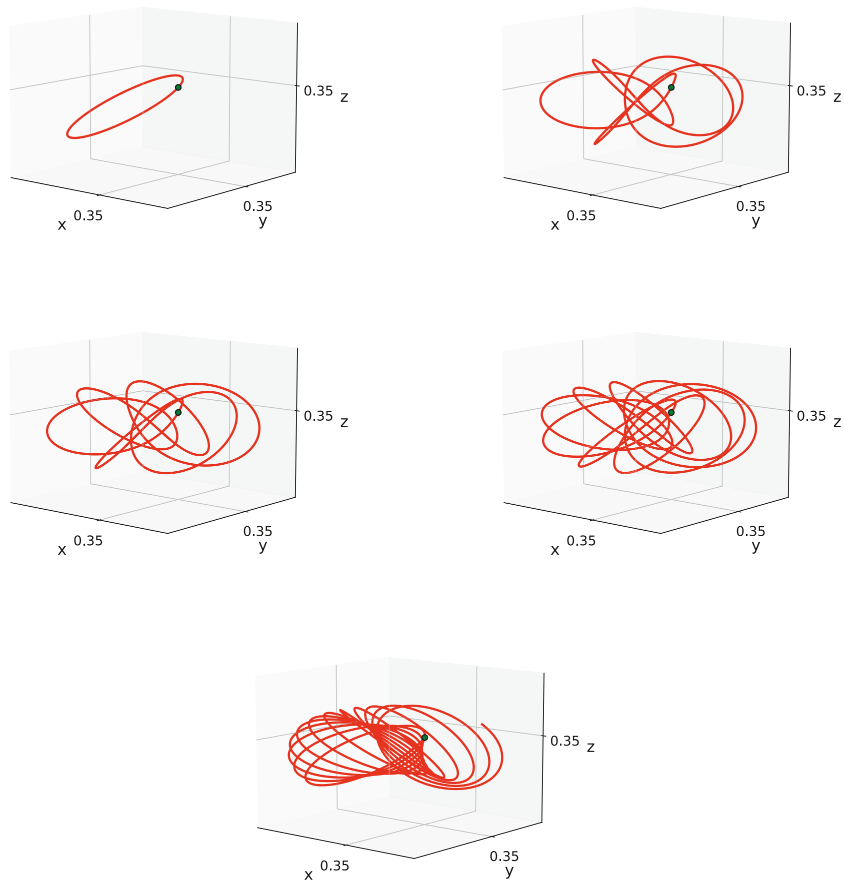

Fig. 4. Magnetic field line (red) at (0.35, 0.35, 0.35) in dimensionless space units. We plot the cases ($m = 3$, $n = 3$), ($m = 4$, $n = 5$), ($m = 5$, $n = 6$), ($m = 8$, $n = 9$) and ($m = 3$, $n = \pi$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where we have defined the parameter λ as the ratio

$$\lambda = \frac{m^2}{n^2}. \tag{20}$$

Eq. (19) can be solved algebraically for δ', obtaining

$$\delta' = \frac{\delta}{2R^2} \pm \frac{1}{2R^2}\frac{\delta^2 + R^2}{\sqrt{\delta^2 + \lambda R^2}}. \tag{21}$$

By making the change of variable $\delta = Rp$, Eq. (21) tranforms into

$$p' = \pm\frac{1}{2R^2}\frac{p^2+1}{\sqrt{p^2+\lambda}}. \tag{22}$$

A further change of variables

$$p = \sqrt{\lambda}\frac{\xi}{\sqrt{1-\xi^2}}, \tag{23}$$

allows us to solve (22) and find the following transcendental equations

$$\pm \ln R = \frac{1}{2}\ln\left(\frac{1+\xi}{1-\xi}\right) + \sqrt{\lambda - 1}\,\arctan\left(\sqrt{\lambda-1}\,\xi\right), \tag{24}$$

for $\lambda \geq 1$, and

$$\pm \ln R = \frac{1}{2}\ln\left(\frac{1+\xi}{1-\xi}\right) - \sqrt{1-\lambda}\,\text{argtanh}\left(\sqrt{1-\lambda}\,\xi\right), \tag{25}$$

for $\lambda \leq 1$.

Eqs. (24) or (25) define the δ function implicitly through the relation

$$\delta = \frac{m}{n}R\frac{\xi}{\sqrt{1-\xi^2}}. \tag{26}$$

4. The topology of the new class

In this section we investigate by numerical methods the new class of solutions found in the previous section. In particular, the topology of the set of magnetic and electric lines of the electromagnetic fields given, at a time $t = 0$, by expressions (13)–(14). A general property is, by construction, that electric and magnetic lines which passes through the same point in the space will be orthogonal.

If the ratio m/n is a rational number, the field lines are closed. As it is shown in Fig. 1 , the particular case when $m = n$ would be another way to find the Hopf fibration and the hopfion. But there is a whole range of new knotted and closed solutions when choosing different values. For example in Fig. 2 we have plotted the case $m = 4$ and $n = 5$.

Moreover, the solutions allow us to take an irrational ratio. In this case, the solutions still are orthogonal but they are not closed anymore. Instead, they describe a torus shape surface. In Fig. 3 we have drawn the particular case $m = \sqrt{3}$ and $n = 2$.

In Fig. 4 we have drawn only a magnetic field line and plotted a sequence of cases to clearly demonstrate how we can change the topology of the field lines controlling the value of the m/n ratio. When the ratio becomes irrational, the line is not closed anymore.

5. Conclusions

In this work, we have studied the possibility of finding electromagnetic fields that are exact solutions of Maxwell equations in vacuum and have the following properties at a fixed instant of time $t = 0$: (i) the magnetic and electric fields can be obtained from two complex scalar fields $\phi(\mathbf{r})$ and $\theta(\mathbf{r})$ through the Rañada's formulation (1)–(2) of electromagnetic knots, and (ii) the solutions satisfy null conditions $\mathbf{E} \cdot \mathbf{B} = 0$ and $E^2 - c^2 B^2 = 0$. Our solutions generalize the hopfion introduced by A. F. Rañada in [1] and provide new examples of electromagnetic knots [2] with different field line topologies. The new class includes Seifert fibrations as well as ergodic cases.

The complex scalar fields ϕ and θ are similar to the ones used in [4], that give field lines that form Seifert fibrations not conserved in time. In order to satisfy the null field conditions, that will allow the structure to be conserved, we have generalized these complex fields by making them dependent on two real numbers m and n and a function $\delta(r^2)$. Using Rañada's formulation (1)–(2) and the null field conditions, we have found a highly nonlinear Eq. (19) that depends on $(m/n)^2$ to be satisfied by the function $\delta(r^2)$.

We have found exact and general solutions for the characteristic Eq. (19). The numerical investigation of the field lines found in this new class of electromagnetic knot solutions shows that closed lines forming torus knots are obtained by taking m and n to be integer and coprime. Moreover, other kind of topologies, in which the field lines are not closed but describe a torus surface, are included into this solutions by taking m/n to be irrational.

The electromagnetic knot type of new solutions found in this work will allow us to understand some interesting physical features of optics and electromagnetism, as the relation between electromagnetic helicity and the photon content with field line topology [6–9].

Dedication to Antonio F. Rañada

The influence of A. F. Rañada on the application of topology to electromagnetism is difficult to overestimate. Our present work largely builds on the achievements made by him during many years of his active research. We had the pleasure of working with him and retain very fond memories of Antonio's commitment and insight in the field he genuinely loved.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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